# SEMI-CLASSICAL BACKREACTION OF QUANTUM SCALAR FIELDS ON AN EVOLVING SPACETIME

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## ABSTRACT

Dillon N. Morse: Semi-Classical Backreaction of Quantum Scalar Fields on an Evolving Spacetime
(Under the direction of Laura Mersini-Houghton)

The formalism required to extend quantum field theory to curved spacetimes has been studied extensively for the last 40 years. In all treatments of the subject the quantum field is heavily influenced by the spacetime curvature; the geometry, however, is assumed to evolve subject only to classical matter and independently of the energies and pressures of the quantum field. The primary obstacle to solving the fully self-consistent backreaction problem lies in the complexity of the (formally divergent) energy momentum tensor of the quantum field. We explore a number of new approaches towards understanding the underlying mathematics in hopes of determining a physically realistic, finite expression for the energy momentum tensor which describes the quantum field in a particular vacuum state while also influencing the evolution of the spacetime geometry itself.

To my parents whose years of love, support, and sacrifice made this possible.

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## LIST OF ABBREVIATIONS AND SYMBOLS

| | |
|---|---|
| FLRW | Friedmann-Lemaitre-Robertson-Walker |
| $\propto$ | Proportional to |
| $\equiv$ | Defined to be equal to |
| $\approx$ | Approximately equal to |
| $\hbar$,$c$, $G$ | reduced Planck's constant, speed of light, and Newton's gravitational constant |
| $\dagger$ | Hermitian adjoint |
| $\mathscr{M}, \mathscr{L}$ | spacetime manifolds |
| $\langle , \rangle$ | inner product |
| $,\quad ;$ | partial and covariant derivative, respectively |
| $[,]$, $(,)$ | denoting anti-symmetrization or symmetrization over a set of indices |
| $\Sigma$ | three-dimensional hypersurface defining a submanifold |
| $T_p$ | tangent space at a point $p$ |
| $\mu$, $\nu$, $\ldots$ | spacetime indices, running from 0 to 3 (e.g. $t, r, \theta, \phi$) |
| **g**, $g_{\mu\nu}$, $g$ | spacetime metric on $\mathcal{M}$ with "mostly positive" signature (-,+,+,+), it's components, and its determinant, respectively |
| $i$, $j$, ... | spacetime indices defining the spacial sector, running from 1 to 3 (e.g. $r, \theta, \phi$ ) |
| $a$, $b$, ... | spacetime indices defining the base space, running from 0 to 1 (e.g. $t, r$) |
| $x^\mu$ | spacetime coordinate |
| $ds^2$ | spacetime interval |
| $dx^\mu$ | differential one-form in the $x^\mu$ direction |
| $h^\mu{}_\nu$ | projection operator to a 3-dimensional submanifold |
| **N**, $N^\mu$ | a vector and its components, respectively |
| **n**, $n_\mu$ | a differential one form and its components, respectively |
| $d\Omega^2$ | metric on the unit two-sphere |
| $\partial_{x^\mu}$ | unit vector in the $x^\mu$ direction |
| $T^{\mu\nu}$, $\tau^{\mu\nu}$ | energy momentum tensors |
| $G^{\mu\nu}$ | Einstein tensor |
| $\epsilon^{\mu\nu}$ | The Levi-Civita totally-antisymmetric pseudo-tensor |

| | |
|---|---|
| $D(x, x')$ | Van-Vleck Morrette determinant |
| $\phi$, $\lambda$, $\psi$, $R$ | undetermined metric functions |
| $\nabla_\mu$, $\nabla^2$ | covariant derivative in the direction of $x^\mu$ and the Laplacian $\nabla_\mu \nabla^\mu$ |
| $\Box$ | d'Alembert operator |
| $R_{\mu\nu}$, $R$ | Ricci tensor and Ricci scalar |
| $C^{\mu\nu\alpha\beta}$, $C$ | Weyl curvature tensor and scalar, respectively |
| $\mathscr{I}^\pm$ | future and past null-infinity, respectively |
| $i^\pm$ | future and past timelike infinity, respectively |
| $i^0$ | spacelike infinity |
| $S$ | action functional |
| $\xi$ | dimensionless coupling constant |
| $u_k$ | mode solutions to the Euler Lagrange equations of motion |
| $a_k$ | lowering operator |
| $\varphi$ | background field |
| $W[J]$ | quantum generating functional |
| $\Gamma[\varphi]$ | quantum effective action |

# Chapter 1: Introduction

This dissertation will center on the analysis of quantum fields in the vicinity of matter which is collapsing to form a black hole. I approach this from a theoretical standpoint, looking to help lay the mathematical foundation necessary to construct simple models which can be described analytically without the need for computer simulations or numerical approximations. The question we hope to answer is whether or not quantum mechanics, which governs micro-scale processes, will have any macro-scale effects on the formation and subsequent evolution of a black hole. While the question we are asking is very general, and we are certainly not the first to ask it, my goals are much more humble: I focus only on trying to model the behavior of quantum fields in the region of a collapsing star, leaving the complex task of describing the behavior of the star-plus-field system to future researchers. While questions of this nature have been addressed for more than forty years, recent investigations [33, 35] have lead to a re-stimulation of discussion in this area [34].

Albert Einstein was the first to show that spacetime wasn't fixed but instead governed by a set of dynamical equations which allow matter to affect the evolution of spacetime just as the curvature of spacetime governs the behavior of the matter. Because of the circular nature of this relationship it is necessary to model both entities, the distribution of matter and energy as well as the background spacetime, simultaneously to get a fully self-consistent description of the system. This is difficult, often impossible, unless many assumptions and approximations are made.

In the late 1960's Leonard Parker pioneered a new area of physics, that of quantum field theory in curved spacetime, when he discovered the phenomena of quantum particle production due to a changing gravitational source. Then in 1975 Stephen Hawking investigated the behavior of quantum fields evolving near a black hole, finding that such a situation inevitably results in a uniform flow of energy radially outwards from the black hole. This energy flux is often referred to as "Hawking radiation" and depends only on the mass of the black hole from which it comes. By a simple argument invoking conservation of energy it must be true that the black hole is losing mass in order to fuel this energy flow, which in turn causes the black hole to shrink in size. Thus black holes evaporate.

Professor Hawking's result, and nearly all results that followed by other physicists modeling similar situations, assume that the spacetime is known in advance and is not influenced by the existence of quantum fields. In this way they treat spacetime as the canvas on which quantum field theory is done, never completing the circle to allow the energy those fields contain to alter the curvature of spacetime in their vicinity. All calculations show the same thing: a quantum field which begins with zero energy (so, a field in its "vacuum-state") will be excited by the changing curvature of spacetime which causes it to carry energy and momentum and therefore to influence the local curvature of spacetime, thus all such models are internally inconsistent in this sense. Even the simplest models which begin in a Schwarzschild spacetime must be inherently flawed from the outset as such models rely on the quantum field containing zero energy which it cannot do once a black hole begins to form. A more realistic model must allow the spacetime to deviate away from the traditional, purely relativistic, solution as these fields are excited and begin to carry energy outwards.

In order to construct a complete and totally self-consistent model it is then necessary to allow the spacetime to evolve in concert with the quantum field. This means that the spacetime cannot be totally specified from the outset as was done by Professor Hawking. A spherically symmetric spacetime can be described by three independent functions each of three spatial coordinates and one time coordinate. With this in mind it is possible to state more concretely the goal of this project: *determine the nature of the energy and momentum of a (simple) quantum field in terms of these three, a-priori unknown, functions which determine the underlying spacetime.* Ultimately we would like to use Einstein's field equations to then determine these three functions, and thus fully describe the behavior of a distribution of matter as it collapses to form a black hole, however this task comes with its own host of complications and lies outside the scope of our current investigation. I do not claim to have computed the desired energy momentum tensor in this dissertation, however I have layed much of the mathematical groundwork necessary towards completing such a calculation.

Concrete calculations in quantum field theory come with a particularly nasty complication:

the values of observable quantities are nearly always infinite if treated naively. Of course this is not physical. To remedy this physicists have developed a set of techniques dubbed “regularization” and “renormalization.” These are methods for removing the infinities in a systematic way in order to get at the underlying physics. This becomes our primary obstacle: how to correctly renormalize quantities when working in a curved, un-specified background spacetime?

Julian Schwinger developed a formalism for calculations in quantum field theory which was then generalized to quantum fields in curved spacetimes by Bryce DeWitt. One result of this work is the Schwinger-DeWitt expansion for the Green’s function, a fundamental object for calculations in field theories, which is valid in any spacetime. While this a very attractive tool for our calculation it turns out to be inadequate. I propose a modification to this calculation technique to recover missing information due to a set of simplifying-assumptions that were made by DeWitt. Such modifications have been studied before, for example in the context of quantum gravity theories (which should be contrasted with all the above discussion which involves quantum fields *in* gravity, an important distinction).

Another approach which has shown promise is in the extension of a previous result involving the study of the conservation equations. Any quantum field should obey the conservation of momentum, energy, etc. (appropriately generalized to general relativity), these impose a set of constraint equations on the observable quantities we hope to compute. Coupled with some input from the Schwinger-DeWitt method (which conspires to allow for the complete, non-series calculation of the trace of a certain matrix) and a generalized symmetry assumption there results a complete set of dynamical equations for all the unknown functions needed to describe the behavior of the quantum field. While the result of this route of inquiry is a formally-complete solution, it is sufficiently complicated that it currently lacks a simple physical interpretation.

In the following chapters the mathematics of general relativity, simplified to the case of a spherically-symmetric spacetime, are presented. I outline the calculation for the matching of two spacetimes along a common (and dynamic) border for this highly-symmetric case. In particular I provide a novel expression for, and discussion of, the mathematics and procedure necessary for piecing together a complete solution in the back-reaction problem. This result is general enough that it can be used to patch together different coordinate charts in to a complete geometrical atlas as may be necessary when describing the spacetime geometry of regions interior and exterior of a collapsing star.

Next the mathematics of quantum field theory in curved spacetime are recapped with a particular emphasis on the formal derivations of quantities which will later be integral for the computation, and subsequent renormalization of, an energy momentum tensor for a quantum scalar field. Here the general theory of semi-classical gravity is outlined as well as a calculation for the effective action in curved spacetime. This chapter lays much of the groundwork for the primary results to be presented in later chapter.

The calculation of the heat kernel has been an ongoing research field for physicists and mathematicians for decades. The main results will be presented along with a modification inspired by some more recent work in the field. The original approximate form of the heat kernel relied on an intrinsically local approximation which was ideal for regularization and the calculation of the form of divergences, however this is inadequate for determining quantum effects which should be global. In order to get at the long-distance behavior of the kernel an alternate form of expansion is suggested. This chapter can be considered a brief literature review of potentially interesting approaches towards solving the problem of calculating the Green's function describing the quantum theory of interest, which ultimately allows for a compuation of the renormalized energy momentum tensor. The primary result I provide here is what I consider the most promising avenue for further explanation: the non-local DeWitt Schwinger expansion which is motivated by similar past work in the field [6, 7].

Central to this problem is the calculation of an energy momentum tensor in a suitably general spacetime. This will be discussed first from a quantum mechanical point of view with an emphasis on the idea of regularization and renormalization. The discussion here is meant to elucidate and clarify the problem, collecting various sources which can guide further work. A modern computer algebra system (Maple) is used to extend the results of one such source, with concrete expressions provided in the appendix.

Following this will be a pair of purely-classical calculations, utilizing only conservation laws in an attempt to constrain the form of such a tensor as much as possible with a minimal of other assumptions. These are among the chief results of this work, extending the approach originally provided by [20] to the case of a completely general sperically-symmetric metric. Before this all known similar concrete calculations relied on a specific choice of a metric which doesn't allow for a completely self-consistent solution.

Units are chosen such that Planck's constant, $\hbar$, and Newton's gravitational constant, $G$, are identically 1 unless otherwise noted. The "mostly-positive" sign convention is adopted wherein the spacetime metric has signature $(-,+,+,+)$, all other conventions follow those of Misner, Thorne, and Wheeler [38].

## Chapter 2: Formalisms in General Relativity

Familiarity with some concepts in General Relativity is assumed, however in order to create a self-contained work it is necessary to present additional formalisms. The goal of this work is to investigate the feasibility of, and contribute to, calculating the energy and momentum content of quantum fields on generic curved spacetime backgrounds. In order to accomplish this goal a certain amount of formalism is required, particularly as it pertains to making statements about physically relevant phenomena without resorting to the use of a particular, concrete spacetimes. That is, whenever possible preference is given to completely covariant approaches.

Despite the stated desire to keep the background spacetime arbitrary, the discussion will be restricted to strictly spherically-symmetric geometries. This simplification provides some hope of allowing for a concrete calculation to be made while still approximating many physically interesting and relevant systems. Particular attention is payed to solutions describing spherically-symmetric matter distributions and black holes. With this in mind, the first section defines a general decomposition of spacetime which takes advantage of the spherical symmetry. Additionally, standard coordinate systems are defined which will be useful when there is need to discuss more concrete physical systems.

The Kodama vector is defined and its utility explained. The Kodama vector is nearly a symmetry of spacetime and comes associated with its own conserved quantity, in the limit of a stationary spacetime the Kodama vector is parallel to the time-like killing vector. It, however, has the benefit of being well-defined even in dynamic spacetimes.

The causal structure of spacetime, which can be visualized locally as "light-cones," can become quite complex. For spacetimes which are asymptotically flat (that is, their curvature tends towards zero for sufficiently large values of a well defined radial coordinate) it is convenient to perform a conformal coordinate transformation which maps the boundaries of spacetime (referred to as future and past infinity) to a finite value. Section two will introduce these concepts formally, demonstrating how these conformal transformations can be used to create diagrammatic representations of spacetime which encode the causal structure.

When constructing physically realizable models of spacetime it is often simplest, or necessary, to isolate individual regions of spacetime from one another and discuss their dynamics independently. For example, when constructing a stellar-collapse model one might investigate the structure of spacetime inside the matter distribution separately from the structure of spacetime outside the matter distribution. It is then necessary to match the two regions of spacetime at their boundaries (along the surface of the matter distribution, in this example) such that together they create a well-defined model of all of spacetime. Section three discusses the requirements that such a matching is well defined, computing the conditions necessary for two spherically-symmetric spacetimes to match along some (non-stationary) border.

## 2.1 Spherical symmetry

The simplification to spherically-symmetric spacetimes will be assumed throughout this work. While perhaps not physically realizable in practice, for many systems of interest the deviations from spherical-symmetry are small. Additionally the goal of this work is to provide insight in to the problem of quantum back-reaction on evolving spacetimes, as such the spherically-symmetric case is a key example for understanding how the mechanisms of quantum field theory will interact with and shape the underlying structure of spacetime.

When considering spherically symmetric spacetimes the line element is taken to be [1]

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = {}^B g_{ab}dx^a dx^b + \bar{R}^2\left(x^a\right){}^F g_{ij}dx^i dx^j \tag{2.1}$$

where the Latin characters $a, b, \ldots$ run over the integers $0, 1$, the Latin characters $i, j, \ldots$ run over the integers $2, 3, \ldots, n$, and $\bar{R}$ is a smooth function of the radial-temporal variables $x^0$ and $x^1$. The base metric ${}^B g_{ab}$ has Lorentzian signature $(-, +)$ while ${}^F g_{ab}$ is the metric on an $(n-2)$-dimensional unit sphere. The plane spanned by the $x^a$ coordinates is considered the "base space." Above every point of the base space there exists a spherical "fiber," namely the unit sphere. Below, the pre factors ${}^B$ and ${}^F$ will be dropped when it is clear from context or from subscripts which space, base or fiber, is being refered to.

When written in this form it is possible to express all geometrical quantities of interest in terms of the three quantities ${}^B g_{ab}$, $\bar{R}$, and ${}^F g_{ij}$. By direct calculation, the nonzero components of the Ricci tensor and the Ricci scalar are given by

$$R_{ab} = \frac{1}{2}{}^B\bar{R}g_{ab} - \frac{2}{\bar{R}\left(x^a\right)}\nabla_a\nabla_b\bar{R}\left(x^a\right) \tag{2.2a}$$

$$R_{ij} = \left(\frac{2m}{r} - \bar{R}\left(x^a\right)\nabla^2\bar{R}\left(x^a\right)\right)g_{ij} \tag{2.2b}$$

$$R = {}^B R - \frac{4}{\bar{R}\left(x^a\right)}\nabla^2\bar{R}\left(x^a\right) + \frac{4m}{\bar{R}^3\left(x^a\right)} \tag{2.2c}$$

where $m = m\left(x^a\right)$ is the Misner-Sharp [37] mass given by $m = \frac{1}{2}\bar{R}\left(x^a\right)\left(1 - \left|\nabla\bar{R}\left(x^a\right)\right|^2\right)$.

While the above affords the freedom to work in a covariant manner it is often convenient to have a concrete expression for the metric. Taking the most general ansatz for the base space in some generic coordinates labeled by $x^0$ and $x^1$ gives

$$ds^2 = -f_1\left(x^0, x^1\right)\left(dx^0\right)^2 + 2f_2\left(x^0, x^1\right)dx^0dx^1 + f_3\left(x^0, x^1\right)\left(dx^1\right)^2 + f_4\left(x^0, x^1\right)d\Omega^2. \quad (2.3)$$

where

$$d\Omega^2 = d\theta^2 + \sin^2(\theta)d\phi^2. \quad (2.4)$$

As general relativity must remain invariant under diffeomorphisms it is possible, without loss of generality, to re-define these coordinates in terms of two new coordinates $t$ and $r$

$$x^0 = x^0(t, r) \quad (2.5a)$$

$$x^1 = x^1(t, r) \quad (2.5b)$$

such that

$$dx^0 = \frac{\partial x^0}{\partial t}dt + \frac{\partial x^0}{\partial r}dr \quad (2.6a)$$

$$dx^1 = \frac{\partial x^1}{\partial t}dt + \frac{\partial x^1}{\partial r}dr. \quad (2.6b)$$

The forms (2.6) can be substituted in to (2.3) then used to define the metric in the new coordinates by picking suitable definitions of the functions (2.5). In this way, for example, the spherically symmetric metric can always be written in coordinates such that the cross-terms $drdt$ vanish. It then takes the form [29, 37]

$$ds^2 = -e^{2\phi(t,r)}dt^2 + e^{2\lambda(t,r)}dr^2 + \bar{R}^2(t, r)d\Omega^2. \quad (2.7)$$

The function $\bar{R}$ has the same interpretation in both (2.1) and (2.7), that of the “areal” radius. That is, construct the integral manifold of the Killing vectors which define the spherical symmetry of the geometry and choose a representative point $x_0^a = (t_0, x_0)$ on this manifold (the submanifold is a two-surface constructed of all points which can be reached from $x_0^a$ via spatial rotations). The proper area of this manifold, which is a scalar under coordinate transformations, is given by $4\pi\bar{R}^2\left(x_0^a\right)$.

At this point only one constraint has been utilized to simplify the form of (2.3) whereas there are two coordinate freedoms available. With the remaining degree of freedom it is possible to re-define any of the three unknown functions. Common choices are $\bar{R}(t,r) = r$ or $\lambda = -\phi$. It should be emphasized that, even after making these choices, the metric is still completely arbitrary (other than the imposed spherical symmetry), the coordinates have just been adapted to make the metric look particularly simple. A choice of coordinates, however, implicitly makes a choice of preferred observers - namely those who remain at constant values of these coordinates.

Schwarzschild geometry is then seen to be nothing more than a special case of (2.7), namely by choosing $\bar{R} = r$, $\phi = -\lambda = \ln(1 - 2M/r)$. Another example of a useful spherically-symmetric spacetime is given by Vaidya [47, 32, 13] as

$$ds^2 = -\left(1 - \frac{2m(v)}{r}\right) dv^2 + 2dvdr + r^2 d\Omega^2 \tag{2.8a}$$

$$ds^2 = -\left(1 - \frac{2m(u)}{r}\right) du^2 - 2dudr + r^2 d\Omega^2. \tag{2.8b}$$

These are often denoted as the ingoing and outgoing Vaidya metrics, respectively. If the function $m$ is taken to be a constant, $m = M$, then both cases reduce to the Schwarzschild metric (in ingoing and outgoing Eddington-Finklestein coordinates, respectively). For physically reasonable scenarios (namely, demanding that the energy density of matter is positive) it follows that $m(v)$ must be a non-decreasing function and $m(u)$ must be non-increasing. Thus (2.8a) is seen to describe the geometry surrounding a black hole or star which is accreting matter (null-dust or radiation) while (2.8b) describes a radiating/evaporating black hole or star.

Working with spacetimes which evolve in time, rather than remaining static, is essential. This means that there will, in general, exist no global timelike Killing vector and thus no natural definition of a time coordinate. This complicates matters significantly as the notion of a positive frequency mode in quantum field theory is defined relative to some time coordinate. Or, in a more geometrical language, a mode function $u_j$ is said to be positive frequency if

$$\mathscr{L}_\zeta u_j = -i\omega u_j, \tag{2.9}$$

for some timelike Killing vector field $\zeta$. Quantum particle creation is then determined by the mixing of positive and negative modes due to gravitational sources.

While there may not exist a timelike Killing vector for spherically symmetric spacetimes there does exist a near analog, the Kodama vector $k^\mu$ defined via

$$k^\mu = {}^B\epsilon^{\mu\nu}\nabla_\nu \bar{R}(x) \tag{2.10}$$

where ${}^B\epsilon^{\mu\nu}$ is the $(1+1)$ dimensional Levi-Civita tensor on the base manifold embedded in to $(3+1)$ dimensions via

$${}^B\epsilon^{\mu\nu} = \begin{pmatrix} \epsilon^{ab} & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.11}$$

The Kodama vector does not reduce to the timelike Killing vector in a static spacetime, however it is parallel to it. In this sense it might be thought of as defining a "time direction." The Kodama vector can be shown to be divergence free, $\nabla_\mu k^\mu = 0$, which implies the existence of a new conserved current

$$J^\mu = G^{\mu\nu}k_\nu. \tag{2.12}$$

That this quantity is covariantly conserved follows from the requirement that the Einstein tensor $G^{\mu\nu}$ be conserved. Statement (2.12) is purely geometrical and follows directly from the decomposition of the metric in the form (2.1) and is thus not a consequence of the Bianchi identities [1]. This "Kodama current" will be exploited by replacing the Einstein tensor in (2.12) with the corresponding energy momentum tensor $T^{\mu\nu}$ via the Einstein equations and demanding that that quantity must also be conserved,

$$0 = \nabla_\mu J^\mu \propto \nabla_\mu \left(T^{\mu\nu}k_\nu\right). \tag{2.13}$$

This is an independent new constraint on the energy momentum tensor which can be used to deduce additional information about its form.

## 2.2 Causal structure of spacetime

In flat spacetime the concept of causality is more easily understood when examining light cones on spacetime diagrams. While the effect of curvature can obfuscate this picture, the general concepts of light cones and null paths still holds. What follows is a convenient construction for obtaining diagrams similar in most respects to the more-familiar spacetime diagrams and which also contain additional information about the complete global structure of the spacetime of interest. Some key terminology is also defined.

Central to this construction is the idea of conformal transformations. These are to be distinguished from coordinate transformations, rather they are mappings which stretch or shrink the entire spacetime geometry via the multiplication of the metric by a 'conformal factor' $\Omega^2(x)$. That is,

$$g^{\mu\nu} \to \bar{g}^{\mu\nu} = \Omega^2(x)\, g^{\mu\nu}. \tag{2.14}$$

Here the function $\Omega(x)$ is taken to be real, non-vanishing, continuous, and finite. This conformal transformation results in a change to the various geometrical quantities of interest such as connection coefficients, curvatures, etc. The result of these under such a transformation can be worked out in a straightforward manner, see e.g. [11].

As an example consider the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{1 - \frac{2M}{r}} dr^2 + r^2 d\Omega^2 \tag{2.15}$$

to which can be applied the series of coordinate transformations [31]

$$\begin{aligned} T &= \left(\frac{r}{2M-1}\right)^{1/2} e^{r/4M} \sinh\left(\frac{t}{4M}\right) \\ X &= \left(\frac{r}{2M-1}\right)^{1/2} e^{r/4M} \cosh\left(\frac{t}{4M}\right) \end{aligned} \tag{2.16a}$$

followed by

$$\begin{aligned} U &= T - X \\ V &= T + X. \end{aligned} \tag{2.16b}$$

The result is then the Schwarzschild metric in the form

$$ds^2 = -\frac{32M^3}{r}e^{r/2M}dUdV + r^2d\Omega^2 \tag{2.17}$$

which is conformal to the metric

$$ds^2 = -dUdV + r^2d\Omega^2 \tag{2.18}$$

which is seen to be the (flat) Minkowksi metric in null coordinates. That is, light rays follow lines of constant $U$ or constant $V$. One key aspect of this example is to also demonstrate that the value $r = 2M$ is not in fact a singularity in the $T, X$ coordinates whereas $r = 0$ is. The former is a *coordinate* singularity in the standard Schwarzschild coordinates, an artifact of that particular coordinate system, while the latter is a true, physical singular point. In modeling a collapsing star and correctly determining the form of the quantum field in the surrounding region even after the classical matter has passed within its Schwarzschild radius $r = 2M$, it will likely be necessary to adopt a similar coordinate system in which the coordinates are well behaved for all $r > 0$.

To the metric (2.18) the transformation

$$\begin{aligned} \bar{u} &= 2\tan^{-1} u \\ \bar{v} &= 2\tan^{-1} v \end{aligned} \tag{2.19}$$

can be applied followed by another conformal transformation to remove the pre-factor. The result is once again a metric of the form (2.18), but this time the coordinates $\bar{u}$ and $\bar{v}$ are only defined on the interval $-\pi \leq \bar{u}, \bar{v} \leq \pi$. The effect has been to shrink in 'infinity' to the values $\bar{u}, \bar{v} = \pm\pi$. Thus the entire spacetime can be represented on a compact diagram. Note that by the nature of the transformations null-rays still correspond to $\bar{u}$ or $\bar{v}$ being constant and thus light-rays still move along $45°$ lines just as in typical spacetime diagrams.

As an example of what such a diagram might look like consider Fig. 1 which illustrates, conceptually, the expected picture of a star collapsing to form a black hole and emitting a flux of positive-energy radiation radially outwards and a balancing flux of negative-energy radiation in to the black hole.

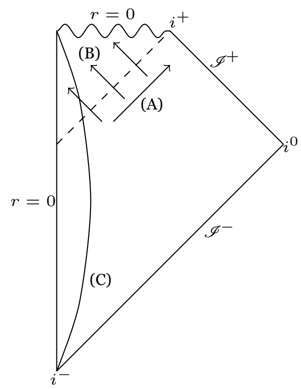


Figure 1: Diagram depicting a collapsing star (C) and formation of an even horizon (dashed line), with an outgoing flux of Hawking radiation (A) originating from near the horizon as well as an ingoing negative-energy flux (B).

The two null lines $\mathscr{I}^{\pm}$ represent future null infinity and past null infinity, respectively. These are the lines on which all null rays terminate. The two points $i^{\pm}$ are the terminal points for all timelike worldlines, in the future and past respectively, whereas $i^0$ represents the terminal point of all spacelike surfaces (surfaces of constant $r$, for example).

## 2.3 Matching of spacetimes along a common boundary

Following the construction in [28], consider an $n$ dimensional manifold $\mathscr{M}$ and an $n-1$ dimensional manifold $\mathscr{L}$ with an imbedding map $\theta : \mathscr{L} \to \mathscr{M}$. Then the image $\theta(\mathscr{L})$ of $\mathscr{L}$ inside $\mathscr{M}$ is an $n-1$ dimensional hypersurface, denoted here to be $\Sigma$. If $p \in \mathscr{L}$ is a point in $\mathscr{L}$ then the map $\theta_* : T_p \to T_{\theta(p)}$, the pushforward, defines a map between tangent spaces. Define $H \equiv \theta_*(T_p)$, an $n-1$ dimensional subspace of $T_{\theta(p)}$.

There exists some nonzero form $\mathbf{n} \in T^*_{\theta(p)}$ such that $\langle \mathbf{n}, \mathbf{X} \rangle = 0$ (or, equivalently, $n_\mu X^\mu = 0$) for all $\mathbf{X} \in H$. This form is unique up to normalization. If the hypersurface $\Sigma$ is given (locally) by the vanishing of some function $f$ such that $df \neq 0$, then (locally) $\mathbf{n} = df$.

Let $\mathbf{g}$ be a Lorentz metric on $\mathscr{M}$ and define the vector $\mathbf{N}$ by $N^\mu = g^{\mu\nu} n_\nu$. By definition $\mathbf{N}$ is orthogonal to all vectors tangent to $\Sigma$, that is $g_{ab} N^a X^b = 0$ for all $\mathbf{X} \in H$. The pullback map, $\theta^*$ can be used to define the metric on $\Sigma$ to be $\theta^* \mathbf{g}$. Consider the three possibilities

1. $g_{\mu\nu} N^\mu N^\nu > 0$, that is $\mathbf{N}$ is a spacelike vector, in this case $\Sigma$ defines a timelike hypersurface within $\mathscr{M}$. Here the metric $\theta^* \mathbf{g}$ is Lorentz.

2. $g_{\mu\nu} N^\mu N^\nu = 0$, that is $\mathbf{N}$ is a null vector, in this case $\Sigma$ defines a null hypersurface within $\mathscr{M}$. Here the metric $\theta^* \mathbf{g}$ is degenerate.

3. $g_{\mu\nu} N^\mu N^\nu < 0$, that is $\mathbf{N}$ is a timelike vector, in this case $\Sigma$ defines a spacelike hypersurface within $\mathscr{M}$. Here the metric $\theta^* \mathbf{g}$ is positive definite.

Assuming a non-null hypersurface $\Sigma$ the normal form $\mathbf{n}$ can be normalized such that $g^{\mu\nu} n_\mu n_\nu = \pm 1$. Now define the tensor $\mathbf{h}$ by

$$h_{\mu\nu} = g_{\mu\nu} \mp n_\mu n_\nu. \tag{2.20}$$

This tensor has the following properties:

$$h_{\mu\nu} X^\mu Y^\nu = g_{\mu\nu} X^\mu Y^\nu \text{ for all } \mathbf{X}, \mathbf{Y} \in H \tag{2.21a}$$

$$h^\mu{}_\nu N^\nu \equiv g^{\mu\gamma} h_{\gamma\nu} N^\nu = 0 \tag{2.21b}$$

$$h^\mu{}_\nu h^\nu{}_\gamma = h^\mu{}_\gamma. \tag{2.21c}$$

Thus $\mathbf{h}$ is a metric on $\theta(\mathscr{L})$ and $h^{\mu}{}_{\nu}$ acts as a projection operator $T_{\theta(p)} \to H$. That is, any vector $\mathbf{X} \in T_{\theta(p)}$ can be written as

$$X^{\mu} = h^{\mu}_{\nu} X^{\nu} \pm n^{\mu} n_{\nu} X^{\nu} \tag{2.22}$$

where the the first term on the right hand side is that part of $\mathbf{X}$ lying in the subspace $H$ and the second term is that part which is orthogonal to $\Sigma$.

Consider any extension $\mathbf{n}$ of the unit normal $\mathbf{n}$ onto some open neighborhood of $\Sigma$ (the extension is necessary so that a derivative can be defined, though the result is independent of extension chosen). Define the tensor $\mathbf{K}$ by

$$K_{\mu\nu} = h^{\alpha}{}_{\mu} h^{\beta}{}_{\nu} \bar{n}_{\alpha;\beta}. \tag{2.23}$$

This is the second fundamental form, or extrinsic curvature tensor, which measures the rate of change of the unit normal $\mathbf{n}$ along the submanifold $\Sigma$.

A typical model of stellar collapse, e.g. [39], will require the splitting of spacetime in to distinct regions. Here it is assumed that there are two such regions defined, to be called the "interior" and "exterior" for the purposes of this section. The two regions must meet along some hypersurface within the larger 3+1 dimensional spacetime, this will be denoted $\Sigma$ and would be, for example, the surface of the (collapsing) star. As in electromagnetism, the (dis)continuity of the fields along a boundary are defined by integration of the Einstein equations across said boundary (just as in the standard "pillbox" integration of Maxwell's equations in electromagnetism). [30]

The result is that the first fundamental form, namely the metric tensor restricted to the hypersurface, must agree on both sides of the boundary whereas the second fundamental form, or extrinsic curvature, may have a discontinuity sourced by a nonzero energy momentum tensor[38]. More concretely,

$$\theta^{*}\mathbf{g}_{\text{interior}} - \theta^{*}\mathbf{g}_{\text{exterior}} = 0 \tag{2.24a}$$

$$K^{\mu}{}_{\nu\text{interior}} - K^{\mu}{}_{\nu\text{exterior}} - \delta^{\mu}{}_{\nu}\text{Tr}\left(K^{\alpha}{}_{\beta\text{interior}} - K^{\alpha}{}_{\beta\text{exterior}}\right) = 8\pi \lim_{\epsilon\to 0}\int_{-\epsilon}^{+\epsilon} T^{\mu}{}_{\nu}\, dn \tag{2.24b}$$

where $n$ is the proper distance measured perpendicularly through $\Sigma$. That is, the two fundamental forms for the submanifold $\Sigma$ are computed using both the interior metric and the exterior one, then set equal as defined in (2.24).

As a concrete example, consider the general spherically symmetric metric (2.7) written in the form

$$ds^2 = -e^{2\phi(t,r)}dt^2 + e^{2\lambda(t,r)}dr^2 + r^2 d\Omega^2. \tag{2.25}$$

Here the final gauge freedom has been used to define the radial coordinate to be equal to the areal radius. Consider the surface of some 'star' to be defined by $r = \tilde{R}(t)$, which is allowed time dependence in order to describe a collapse scenario. At this point it has not been specified whether this metric is describing the 'interior' or 'exterior' geometry as both will have this same form (with different choices of the functions $\phi$ and $\lambda$).

Now consider the submanifold $\Sigma$ defined by $0 = r - \tilde{R}(t)$. Restricting the metric (2.25) to this hypersurface gives

$$ds^2 = -e^{2\phi}\left(1 - e^{2\lambda - 2\phi}\dot{\tilde{R}}^2\right)dt^2 + \tilde{R}d\Omega^2. \tag{2.26}$$

The same metric can be written down for the alternative region and the two set equal to each other. The result is that the terms multiplying each $dt^2$ must be equal, defining one equation for the six unknowns (two $\lambda$'s, two $\phi$'s, and two $\tilde{R}$'s). This equation can be shown to be equivalent to demanding that two observers, one just inside the collapsing star and one just outside, must experience equal amounts of elapsed proper time. The coefficients of the spherical metrics $d\Omega^2$ must also be equal, this leads to the requirement that $\tilde{R}$ for the interior must be equal to $\tilde{R}$ for the exterior, each written in terms of their respective coordinate systems.

# Chapter 3: Quantum Field Theory in Curved Spacetime

Some familiarity with quantum field theory is assumed, however all discussions here will be contained to free scalar theories. In the presence of spacetime curvature even these simple field theories take on complexity. The fields may, for example, scatter off the curvature itself. Even the concept of a vacuum state is ill-defined when you demand that the theory be invariant under diffeomorphisms. Stated differently, two observers need not agree on the particle content (or lack thereof) in a region of spacetime.

In passing to curved spacetimes most of the flat-spacetime constructions of quantum field theory can be preserved with the promotion of partial derivatives to covariant derivatives which depend, locally, on the connection which is here always chosen to be the standard metric-compatible Levi-Civita connection. Additionally, the curvature of spacetime itself provides a new length scale with which to work since the Ricci curvature scalar has units of inverse length squared. This opens up the possibility of adding additional terms to any action functional which describes the theory which can in turn give rise to an explicit coupling between quantized matter fields and spacetime itself. The first and second sections of this chapter will introduce the formalism of quantum field theory in curved spacetime and derive a few of the interesting consequences that arise from the inclusion of non-flat geometries.

The primary difficulty with this work is in gaining a mastery of the mathematical machinery necessary not only to set up a theory, but to properly renormalize it as well as derive from it interesting physical predictions. The remainder of this chapter will address these concerns, presenting an appropriate formalism of field theory and discussing the problem of renormalization and how it differs when spacetime curvature is introduced.

## 3.1 Free scalar field

Throughout we will consider the case of a massive scalar field $\phi$ coupled to spacetime curvature. That is, the the quantum theory governed by

$$S_{\mathrm{QM}} = \int d^4x \, \frac{-1}{2} g^{1/2} \Big[ \nabla_\mu \phi\left(x\right) \nabla^\mu \phi\left(x\right) \tag{3.1}$$

$$+ \left( \frac{m^2}{\hbar^2} + \xi R(x) \right) \phi\left(x\right)^2 \Big] \tag{3.2}$$

where $R$ is the Ricci scalar and $\xi$ is a dimensionless coupling constant. There are two length scales set by this action: that of the background spacetime curvature entering the action through $R$ and that of the scalar field entering through the Compton wavelength $\hbar/m$. In the context of black hole spacetimes the former is directly tied to the mass of the black hole. When varied with respect to the field $\phi$ (3.1) gives rise to the classical equations of motion for $\phi$

$$\left[ \Box - \left( \frac{m^2}{\hbar^2} + \xi R(x) \right) \right] \phi(x) = 0 \tag{3.3}$$

where $\Box = \nabla^\mu \nabla_\mu$ is the usual d'Alembertian operator in curved spacetime.

For a given spacetime, defined by the choice of a particular metric tensor $g_{\mu\nu}$, the equation of motion is explicitly dependent on the spacetime curvature both through the potential $V(x) = \frac{m^2}{\hbar^2} + \xi R(x)$ and through the d'Alembertian. Thus the evolution of quantum fields, and any observables derived from them, are intrinsically linked to the spacetime of choice.

When investigating the effects of such a theory the starting point often involves seeking a complete set of mode solutions $u_k(x)$ out of which all quantities of interest can be constructed. By definition this is an orthonormal family of functions each of which furnishes a solution of (3.3). The field itself is then constructed as

$$\phi(x) = \sum_k a_k u_k + a_k^\dagger u_k^* \tag{3.4}$$

where the index $k$ represents all possible quantities necessary to define the family of modes and the sum formally represents the sum and integral over all such indices. The creation and annihilation operators, $a$ and $a^\dagger$, define the vacuum state and thus a choice of mode-solutions

$u_k$ implicitly corresponds to the choice of a vacuum.

In static spacetime there is a preferred time coordinate with respect to which to define "positive-frequency." In curved spacetime there is not necessarily a timelike Killing vector with respect to which to define a frequency. As such the choice of a preferred family of mode-solutions, and therefore a vacuum state, is ambiguous. Each observer carries with themselves a locally-flat neighborhood with respect to which they define their own natural Minkowski-like vacuum state. There is no expectation that two different observers would agree on such a definition.

To show the mechanism for this behavior consider an alternate expansion of $\phi$ in terms of different mode solutions $\bar{u}_k$ (say, defined to be positive frequency in terms of the time coordinate of some accelerated observer) [12]. Then

$$\phi(x) = \sum_k \bar{a}_k \bar{u}_k + \bar{a}_k^\dagger \bar{u}_k^* \tag{3.5}$$

defines a new set of annihilation and creation operators $\bar{a}_k$ and $\bar{a}_k^\dagger$ and with them a new vacuum state such that $\bar{a}_k |\bar{0}\rangle = 0$ for all $k$. This is the quantum state for which the accelerated observer's particle detector would register zero detections. Since both sets of mode solutions, $u_k$ and $\bar{u}_k$, form a complete set, by definition there must exist an expansion of the form

$$\bar{u}_k = \sum_i \left(\alpha_{ki} u_i + \beta_{ki} u_i^*\right) \tag{3.6}$$

or

$$u_i = \sum_k \left(\alpha_{ki}^* \bar{u}_k - \beta_{ki} \bar{u}_k^*\right). \tag{3.7}$$

These matrices, $\alpha$ and $\beta$, are known as the Bogolubov coefficients and can be determined explicitly via inner products of the form $(\bar{u}_k, u_i)$.

This then gives rise to an expansion of one set of creation/annihilation operators in terms of the other,

$$a_i = \sum_k \left(\alpha_{ki} \bar{a}_k + \beta_{ki}^* \bar{a}_k^\dagger\right) \tag{3.8a}$$

$$\bar{a}_k = \sum_i \left( \alpha^*_{ki} a_i - \beta^*_{ki} a^\dagger_i \right). \tag{3.8b}$$

Consider now the result of the barred annihilation operator acting on the un-barred vacuum state,

$$\bar{a}_k \left| 0 \right\rangle = -\sum_i \beta^*_{ki} a^\dagger_i \left| 0 \right\rangle \neq 0. \tag{3.9}$$

That is, the accelerated observer would detect particles while in the vacuum state of the original, un-barred observer whenever the $\beta$ matrix elements (which connect the negative-frequency $u_j$ modes to the positive frequency $\bar{u}_k$ modes via (3.6)) are nonzero. Any two observers who disagree on the definition of the time coordinate will also disagree on the particle content of the universe.

This mechanism was used to calculate the exact flux of particles that an accelerated observer would measure in [24, 21, 46]. Using the equivalence principle, which roughly states that a non-inertial reference frame should be indistinguishable from a gravitational force, the acceleration of the observer can be replaced by the surface gravity of a massive body to determine the particle creation as observed by a stationary observer near a gravitational source.

The results of this thought experiment are in agreement with the original calculation [27]. Stephen Hawking showed that such a theory, when applied to black hole spacetimes, must give rise to Hawking radiation. That is, for a particular choice of vacuum state a black hole is seen to emit a radial flux of radiation. The state in question is the so-called Unruh state which is defined by choosing plane-wave mode solutions on $\mathcal{I}^-$. This corresponds, physically, to a vacuum state as defined by an inertial observer far from the black hole and long before the onset of collapse. One immediate consequence of this result is that black holes must shrink if energy conservation is to be enforced. While the exact mechanism of such an evaporation procedure has yet to be fully described mathematically in a self-consistent way, it is only reasonable to expect that any energy that escapes to $\mathcal{I}^+$ must be subtracted from the mass of the black hole.

This, then, naturally gives rise to an inconsistency in the model. In order to derive this result it was assumed that spacetime was a vacuum outside the collapsing star and thus described by the Schwarzschild metric. However such a model apparently gives rise to an energy flux towards $\mathcal{I}^+$ which immediately violates the assumption on which the model was constructed:

that spacetime is a vacuum outside the star. Thus such descriptions of a star collapsing in to a black hole and giving rise to Hawking radiation are explicitly self-inconsistent.

## 3.2 Semi-classical gravity

We seek a solution to the two issues described in the preceding paragraphs. This requires a collapse model which manifestly gives rise to energy conservation and black hole evaporation without the need to add it in by hand later, as such it is expected that the collapse and subsequent evaporation are both dependent on the evolution of our scalar field. Second, we require a model which is completely self-consistent. This necessarily predicates a solution in which the exterior region deviates from the standard Schwarzschild spacetime.

When doing quantum field theory in curved spacetime the use of a particular solution of Einstein's equations is seen as specifying the classical background on which the quantum fields evolve. In this way the spacetime curvature directly influences the behavior of the quantum fields and gives rise to unique features not present in flat-spacetime quantum field theory. A complete semi-classical solution completes the circle, allowing the quantum fields themselves to influence the spacetime by solving the Einstein equations with $\phi(x)$ contributing to the source of spacetime curvature.

To these ends we turn to the theory of semi-classical gravity in which the back-reaction of our scalar field will be taken in to account when solving for a spacetime model. Here gravity is treated classically, only the matter fields themselves are quantized.

Einstein's equations can be derived from an action principle by considering

$$S = S_{\text{EH}} + S_{\text{matter}}. \tag{3.10}$$

where $S_{\text{EH}}$ is the Einstein-Hilbert action who's variation with respect to the metric tensor $g_{\mu\nu}$ gives rise to the Einstein tensor and $S_{\text{matter}}$ is the matter-action which, when similarly varied, provides the energy momentum tensor which acts as the source of curvature. In requiring that the variation in the total action go to zero the Einstein field equations are recovered. The matter action will be further decomposed in to two independent parts

$$S_{\text{matter}} = S_{\text{classical}} + S_{\phi}. \tag{3.11}$$

Variation of this matter action then gives rise to an energy momentum tensor which is the sum of two disjoint parts,

$$T^{\mu\nu} = T^{\mu\nu}_{(\text{classical})} + \left\langle \tau^{\mu\nu}_{(\phi)} \right\rangle . \tag{3.12}$$

where $\tau$ is the energy momentum tensor of the scalar field which should be evaluated as an expectation value in a particular quantum state. The fundamental problem of semi-classical gravity then lies in searching for a solution of

$$\frac{1}{8\pi} G^{\mu\nu} = T^{\mu\nu}_{(\text{classical})} + \left\langle \tau^{\mu\nu}_{(\phi)} \right\rangle . \tag{3.13}$$

The decomposition of (3.12) is a result of assuming that the classical and quantum contributions are not explicitly coupled in (3.11), that is that there are no cross-terms in the action. The result of this is that the two matter sources do not directly interact; their influence on one another is only a consequence of their mutual influence on, and reaction to, spacetime curvature through (3.13).

The Einstein tensor, $G^{\mu\nu}$, is covariantly conserved. As a consequence of (3.13) then so must be the total energy momentum tensor. That is,

$$\nabla_\mu \left( T^{\mu\nu}_{(\text{classical})} + \left\langle \tau^{\mu\nu}_{(\phi)} \right\rangle \right) = 0. \tag{3.14}$$

In the region external to any classical matter, where the only source of spacetime curvature is the scalar field $\phi$, it is then clear that $\nabla_\mu \left\langle \tau^{\mu\nu}_{(\phi)} \right\rangle = 0$. However in the interior region, where both types of matter are present, the best that can be said is that the divergence of the classical energy momentum tensor is equal to the negative divergence of the quantum one. To assume that the two tensors are independently conserved is sufficient, but not necessary, to meet the overall requirement. It is a stronger requirement. However, given that the two sources of matter are coupled *only* through their mutual interaction with spacetime curvature it is perhaps not too bad an approximation to demand that each is independently conserved. In this way they can still exchange energy and momentum, though only by using spacetime as an intermediary.

A complete, self-consistent, solution of the semi-classical Einstein equations which takes in to account the backreaction of quantum fields relies, ultimately, on the ability to compute the energy-momentum tensor $\langle \tau \rangle$ as a functional of the as-yet-undetermined metric functions.

With this in hand, (3.13) furnishes a system of differential equations with both curvature and source written in terms of the metric. Thus the problem can be broken down in to two parts: first we must compute an expression for the expectation value of the energy momentum tensor that is valid for a family of metrics rather than just a single particular spacetime, and second we must solve (3.13) in order to select a particular metric from that family which obeys the desired boundary conditions (namely by specifying the quantum state of the scalar field). Here we focus primarily on the first step.

The variation of (3.1) with respect to $g_{\mu\nu}$ gives

$$
\begin{aligned}
\tau^{\mu\nu} =& \nabla^{\mu}\phi\nabla^{\nu}\phi - \frac{1}{2}g^{\mu\nu}\nabla^{\rho}\phi\nabla_{\rho}\phi \\
&+ \frac{1}{2}g^{\mu\nu}m^2\phi^2 - \xi\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right)\phi^2 \\
&+ \xi\left[g^{\mu\nu}\Box\left(\phi^2\right) - \nabla^{\mu}\nabla^{\nu}\left(\phi^2\right)\right]
\end{aligned}
\tag{3.15}
$$

which, together with (3.4), allows for a naive computation of the $\langle\tau^{\mu\nu}\rangle$. This expression is, however, ultraviolet divergent. Furthermore, this divergence is intrinsic to the curved spacetime in the sense that it cannot be removed by simply subtracting an equivalent, and also divergent Minkowski, “zero-point” energy-momentum tensor.

The properly regularized and renormalized expectation value of (3.15) has been computed in a few very limited circumstances, for example in [16]. In this case attention is restricted to the static Schwarzschild spacetime, in particular the only region in which an exact value for the energy momentum tensor is calculated is near, but external to, the event horizon as well as near infinity. This is because the chosen calculation method depends intrinsically on arguments involving the form of the tensor in various different quantum states and utilizes the fact that the divergent terms are state-independent. The computation of $\langle\tau^{\mu\nu}\rangle$ in any dynamic spacetime or for a general family of spacetimes remains an open problem.

## 3.3 The effective action

This derivation follows closely those found in [42, 45].

Consider the action

$$\begin{aligned} S[\phi] &= -\frac{1}{2}\int dV_x dV_{x'} A_{IJ}\left(x,x'\right)\phi^I(x)\,\phi^J\left(x'\right) + \int dV_x J_I(x)\,\phi^I(x) \\ &= -\frac{1}{2}A_{ij}\phi^i\phi^j + J_i\phi^i \end{aligned} \tag{3.16}$$

which serves to define the condensed notation that will be used throughout. Here the unspecified field $\phi^I$ is coupled to an external source $J_I$ but otherwise has no additional interactions. In this notation the delta function can be expressed as

$$\delta^i_j\phi^j = \int dV_{x'}\delta^I_J\delta\left(x,x'\right)\phi^J\left(x'\right) \tag{3.17}$$

where $\delta^I_J$ is the Kronecker delta and $\delta(x,x')$ is the standard bi-function Dirac delta distribution. As an example, a free scalar field coupled to the Ricci scalar can be derived from

$$A_{ij} = \left(\Box_x + m^2 + \xi R\right)\delta\left(x,x'\right)\delta_{IJ}. \tag{3.18}$$

The classical equation of motion for the field $\phi$ can be derived from the principle of stationary action as

$$\frac{\delta S[\phi]}{\delta\phi^j} = 0 \tag{3.19}$$

which leads to

$$A_{ij}\phi^i = J_j. \tag{3.20}$$

This leads to the (Feynman) Green's function $G_F^{ij}$ defined via

$$A_{ij}G_F^{jk} = \delta_i^{\ k} \tag{3.21}$$

and gives a formal solution to (3.20)

$$\phi^k = G_F^{jk}J_j. \tag{3.22}$$

Consider two spacelike hypersurfaces $\Sigma_1$ and $\Sigma_2$ where all points of $\Sigma_2$ lie to the future of $\Sigma_1$. Given a complete set of commuting observables $\zeta_{1,2}$ defined on $\Sigma_{1,2}$, respectively, we can define two quantum states $|\zeta_1', \Sigma_1\rangle$ and $|\zeta_2', \Sigma_2\rangle$ such that $\zeta_{1,2}\,|\zeta_1', \Sigma_1\rangle = \zeta_{1,2}'\,|\zeta_1', \Sigma_1\rangle$. These two states are related by a unitary transformation $U_{12}$ such that

$$\left|\zeta_2', \Sigma_2\right\rangle = U_{21}\left|\zeta_1', \Sigma_1\right\rangle \tag{3.23}$$

or

$$\left\langle \zeta_2', \Sigma_2\right|\left.\zeta_1', \Sigma_1\right\rangle = \left\langle \zeta_1', \Sigma_1\left|U_{12}^{-1}\right|\zeta_1', \Sigma_1\right\rangle. \tag{3.24}$$

Now this unitary operator depends on the choice of commuting sets of observables as well as on the choice of hypersurfaces on which to define them, a change in these induces a change in $U$ which leads to

$$\delta\left\langle \zeta_2', \Sigma_2\right|\left.\zeta_1', \Sigma_1\right\rangle = \left\langle \zeta_1', \Sigma_1\left|\delta U_{12}^{-1}\right|\zeta_1', \Sigma_1\right\rangle. \tag{3.25}$$

The operator $U$ can then be expressed in terms of a Hermitian operator $S$ via

$$U_{12} = e^{-\frac{i}{\hbar}S_{12}} \tag{3.26}$$

which gives rise to

$$\delta U_{12}^{-1} = \frac{i}{\hbar}U_{12}^{-1}\delta S_{12}. \tag{3.27}$$

Inserting this in to (3.25) and allowing $U_{12}^{-1}$ to act on the bra to its left gives

$$\delta\left\langle \zeta_2', \Sigma_2\right|\left.\zeta_1', \Sigma_1\right\rangle = \frac{i}{\hbar}\left\langle \zeta_1', \Sigma_1\left|\delta S_{12}\right|\zeta_1', \Sigma_1\right\rangle \tag{3.28}$$

which is nothing more than a statement of Schwinger's variational principal[44]. That is, the variation in the transition amplitude between two states is given by matrix amplitudes of the variation of the action. For ease of notation I will often use the notation $|\text{in/out}\rangle$ to refer to these two quantum states.

Define the generating functional $W[J]$ via

$$\langle \text{out}|\,\text{in}\rangle = e^{\frac{i}{\hbar}W[J]} \tag{3.29}$$

so that

$$\frac{i}{\hbar}\delta W[J]\langle \text{out}|\,\text{in}\rangle = \delta\langle \text{out}|\,\text{in}\rangle = \frac{i}{\hbar}\langle \text{out}\,|\delta S|\,\text{in}\rangle\,. \tag{3.30}$$

If this variation is taken with respect to the external source $J$ this gives

$$\begin{aligned}\frac{\delta W[J]}{\delta J_i} &= \langle \text{out}|\,\text{in}\rangle^{-1}\left\langle \text{out}\left|\frac{\delta S}{\delta J_i}\right|\text{in}\right\rangle \\ &= \langle \text{out}|\,\text{in}\rangle^{-1}\left\langle \text{out}\left|\phi^i\right|\text{in}\right\rangle \\ &= \left\langle \phi^i\right\rangle \end{aligned} \tag{3.31}$$

where I have introduced the notation

$$\langle A\rangle \equiv \frac{\langle \text{out}\,|A|\,\text{in}\rangle}{\langle \text{out}|\,\text{in}\rangle}. \tag{3.32}$$

Note that this is not a true expectation value as it is given by an off-diagonal matrix element whenever the states 'out' and 'in' differ from one another. Combining (3.31) and (3.22) and using the fact that $G$ and $J$ are c-numbers

$$\frac{\delta W[J]}{\delta J_i} = G_F^{ij}J_j. \tag{3.33}$$

Note that information about the state is now contained in the Green's function $G^{ij}$. Integration then yields

$$W[J] = \frac{1}{2}J_i G_F^{ij} J_j + W[0]\,. \tag{3.34}$$

The generating functional $W$ is then determined up to a constant.

In order to determine this constant consider another variation, this time with respect to $A_{ij}$. From (3.16) and (3.30) this gives rise to

$$\delta W[J] = -\frac{1}{2}\delta A_{ij}\left\langle T\phi^i\phi^j\right\rangle \tag{3.35}$$

where $T$ here refers to the time-ordering operator. It is also possible to generate two copies of the field $\phi$ by varying the left-hand side of (3.31) again with respect to $J$ evaluated at a different spacetime point, this gives

$$\frac{\delta^2 \langle \text{out} | \text{in} \rangle}{\delta J_i \delta J_j} = \left( \frac{i}{\hbar} \right)^2 \langle \text{out} \left| T \phi^i \phi^j \right| \text{in} \rangle . \tag{3.36}$$

Using (3.29) and (3.33) the left hand side of this expression is given by

$$\frac{i}{\hbar} \frac{\delta^2 W [J]}{\delta J_i \delta J_j} e^{\frac{i}{\hbar} W[J]} + \left( \frac{i}{\hbar} \right)^2 G_F^{ik} J_k G_F^{jl} J_l e^{\frac{i}{\hbar} W[J]} .$$

Setting $J = 0$ then gives rise to

$$\left. \frac{\delta^2 W [J]}{\delta J_i \delta J_j} \right|_{J=0} = \frac{i}{\hbar} e^{-\frac{i}{\hbar} W[0]} \langle \text{out} \left| T \phi^i \phi^j \right| \text{in} \rangle \tag{3.37}$$

however from (3.34) the second derivative of $W$ is given by $G$ (note that by (3.21) $G$ has no dependence on $J$), thus

$$G_F^{ij} = \frac{i}{\hbar} \langle T \phi^i \phi^j \rangle . \tag{3.38}$$

Inserting this back in to (3.35) and using the formal property $G = A^{-1}$ then gives

$$\frac{i\hbar}{2} \text{Tr} \left( A^{-1} \delta A \right) = \frac{i\hbar}{2} G^{ij} \delta A_{ij} = \delta W [0] . \tag{3.39}$$

The left hand side of this expression can be manipulated using

$$\text{Tr} \left( A^{-1} \delta A \right) = \text{Tr} \left( \delta \ln \left( \ell^2 A \right) \right) = \delta \text{Tr} \left( \ln \left( \ell^2 A \right) \right) = \delta \ln \left( \det \left( \ell^2 A \right) \right)$$

where a constant $\ell$ with dimensions of length was inserted to keep the argument of the logarithm dimensionless. This gives

$$W [J] = \frac{1}{2} J_i G_F^{ij} J_j + \frac{i\hbar}{2} \ln \left( \det \left( \ell^{-2} G_F^{ij} \right) \right) . \tag{3.40}$$

Define now the background field $\varphi^i$ by

$$\varphi^i = \langle \phi^i \rangle = \frac{\delta W [J]}{\delta J_i} \tag{3.41}$$

and use this to perform a Legendre transformation on the generating functional $W$ via

$$\Gamma\left[\varphi\right] \equiv W\left[J\right] - J_i \varphi^i \tag{3.42}$$

which defines the "effective action" $\Gamma$. Contracting this expression with the appropriate 'in' and 'out' states shows that $\varphi^i$ also satisfies (3.20), using this to eliminate the external source $J$ while using (3.40) in (3.42) gives a final expression for the effective action of a free field

$$\Gamma\left[\varphi\right] = S\left[\varphi\right] + \frac{i\hbar}{2} \ln\left(\det\left(\ell^2 A_{ij}\right)\right). \tag{3.43}$$

Thus the effective action for a free field is composed of the classical action plus a "quantum correction" term. This correction term is known as the 'one-loop' effective action, the explicit presence of $\hbar$ in this term denoting that it is the first order term in a perturbative expansion for all quantum corrections to the classical action. For a free field all higher order terms vanish and this expression is exact. The correction term can be thought of as governing the fluctuations about the background classical field $\varphi$.

## 3.4 Regularization and renormalization

The expression (3.43) for the effective action of a free field is divergent. The precise form of the divergence is dependent on the theory, that is on the form of $A_{ij}$ which contains all relevant information. While a completely self-contained derivation of the divergences goes beyond the scope of this work, it is advantageous to outline a particular derivation courtesy of [42] which will make connection with the Heat Kernel to be discussed in the next chapter.

The goal of this section is to explore the one-loop term in the effective action,

$$W[0] = \frac{i\hbar}{2} \ln\left(\det\left(\ell^2 A_{ij}\right)\right). \tag{3.44}$$

Consider first the operator $A_{ij}$ and solutions $f_n^j$ of the eigenvalue problem

$$A_{ij} {f^j}_n = \lambda_n f_{ni} \tag{3.45}$$

such that the ${f^j}_n$ are normalized according to ${f^j}_n f_{mj} = \delta_{nm}$ (using the condensed notation defined in the previous section).

Consider a small change $\delta A$ in the operator $A$ and the corresponding variation in $W[0]$

$$\delta W[0] = \frac{i\hbar}{2}\delta\left(\mathrm{tr}\ln A\right) = \frac{i\hbar}{2}\mathrm{tr}\left(A^{-1}\delta A\right). \tag{3.46}$$

Using the formal identity for the inverse operator

$$A^{-1} = i\int_0^\infty ds\; e^{-isA} \tag{3.47}$$

which assumes that $A$ has a small negative imaginary part (which is equivalent to choosing the conditions for the Feynman propagator). It is seen that

$$\delta W[0] = -\frac{i\hbar}{2}\delta\int_0^\infty \frac{ds}{s}\mathrm{tr}e^{-isA}. \tag{3.48}$$

This then implies

$$W[0] = -\frac{i\hbar}{2}\int_0^\infty \frac{ds}{s}\mathrm{tr}e^{-isA} = -\frac{i\hbar}{2}\int_0^\infty \frac{ds}{s}\sum_n e^{-is\lambda_n} \tag{3.49}$$

where the last equality uses the eigenvalue construction above ($A$ is diagonalizable) as well as the Taylor series definition of the matrix exponential. Now define the 'heat kernel'

$$K^i_j(s,x,y) = \sum_n e^{-is\lambda_n} f(x)^i_n f(y)^*_{nj}. \tag{3.50}$$

Then the expression (3.49) corresponds to the coincidence-limit, $y \to x$, of $K$ via

$$W[0] = -\frac{i\hbar}{2}\int_0^\infty \frac{ds}{s}\int dV_x\ K^j_j(s,x,x) \tag{3.51}$$

which uses the normalization property of the eigenfunctions. Note that, via it's definition, the kernel $K$ satisfies an analog of the Schrodinger equation

$$i\frac{\partial}{\partial s}K^i_j(s,x,y) = A^i_k K^k_j(s,x,y) \tag{3.52}$$

with $s$ playing the role of time. For this reason $s$ is often called the 'proper time' despite the fact that it must have units of inverse length squared. Note that, because of the definition (3.50), the kernel satisfies the boundary condition

$$\lim_{s\to 0} K^j_j(s,x,y) = \delta^i_j\delta(x,y). \tag{3.53}$$

From this and (3.51) it is clear that $W[0]$ diverges on the lower-limit of the $s$ integral. The kernel $K$ can be shown to have an asymptotic expansion of the form

$$K(s,x,x) \propto \frac{1}{(is)^2}\sum_{k=0}^{\infty}(is)^k E_k(x) \tag{3.54}$$

for small $s$. Further it can be shown that the divergences in $W[0]$, and thus in the effective action, correspond to the first $n/2+1$ terms in this expansion (where $n$ is the dimension of the spacetime).

This process corresponds to a regularization of the one-loop effective action. In order to properly renormalize the action these divergent terms must be removed. In order to do this it is advantageous to include counterterms to the classical action corresponding to these divergent terms, with un-determined bare coupling constants. The coupling constants are then allowed to absorb the infinite pre-factor in the diverging terms. In order to accomplish this it is necessary to add additional higher-order-in-derivative terms to the classical Einstein-Hilbert action. These terms may be taken seriously as alterations to Einstein's gravity theory, or they may be removed by subsequently setting the new coupling constants equal to zero.

# Chapter 4: The Heat Kernel

The standard prescription for calculating the Green's function for the scalar field theory (3.1) can be shown to reduce to solving an $n+1$ dimensional 'Schrodinger' equation (3.52) for the heat kernel. Beyond field theory, explorations of properties of the heat kernel permeate mathematics. The addition of curved spacetime geometry, however, greatly complicates matters. There are only a limited number of approaches that have been explored in this context, few of which show promise of being useful for solving the problem at hand.

The original derivation by DeWitt, motivated by the proper time method of Schwinger, lead to an asymptotic expansion for $K$ which leads to an expression which can be written in terms of Hankel functions. This is implicitly a summation in inverse powers of the field mass $m$, and thus this asymptotic series has hope of converging only in the limit $M/m << 1$, where $M$ is the mass of any spherical matter distribution we care to model and which will implicitly set the scale of curvature in this problem. In addition to questions of convergence, this expansion only provides knowledge of $K(s, x, x')$ in the small-$s$ regime. The Feynman Green's function, however, involves the integral of $K$ over all positive values of $s$. It will be shown that $K$ has the alternate interpretation as the probability amplitude for a particle to propagate from point $x$ to point $x'$ in 'time' $s$. Thus the Schwinger-DeWitt technique is intrinsically limited to explorations of the local-structure of the theory. It can only depend on curvature in a neighborhood of point $x$. Quantum behaviors such as particle creation, however, are intrinsically global in nature and will depend on the quantum state initially chosen for the field. This asymptotic expansion is completely state-independent. While it may not be useful for calculating the quantum effects we desire, it is however crucial to understand the ultraviolet divergences of the theory. It is in this manner that the Schwinger-DeWitt technique finds utility.

The latter section of this chapter provides a brief overview of alternative techniques for calculating $K$. While most attempt to provide an alternative expansion or perturbation scheme in which to calculate the kernel, one technique due to Leonard Parker takes the approach of solving the ‘Schrodinger’ equation (3.52) via a path integral technique. This can then be approximated by a series of Gaussian integrals dependent on paths which lie ‘near’ some extremal path. Another promising approach results in alternative expansion in powers of $1/s$ instead of $s$, allowing the global structure of spacetime to be probed. This allows for approximations to be made for the kernel at both large and small scales, which then need to be joined in some way at intermediate length scales.

## 4.1 The Schwinger-DeWitt formulation

An alternate derivation of the heat kernel $K$ (3.50) is provided which helps elucidate its interpretation. The Schwinger-DeWitt approximation for determining $K$ as an asymptotic expansion is also outlined with an emphasis on its utility and its shortcomings. The derivation below follows closely that which can be found in [18, 23]. Much of the formalism here is an expansion to curved spacetime of Schwinger's proper-time method for finding the Feynman Green's function $G_F(x, x')$ [43]. Here we are most interested in finding the Hadamard Green's function $G^{(1)}(x, x')$ which can be related to $G_F$ via

$$G_F(x, x') = \bar{G}(x, y) - \frac{1}{2} i G^{(1)}(x, y) \tag{4.1}$$

where $\bar{G}$ is the principal-value function which is equal to one-half the sum of the advanced and retarded Green's functions.

We define the differential operator $F(x, x')$ by

$$\int dV_x F(x, x')\phi(x') \equiv g(x)^{1/2} \left[\Box_x - \frac{m^2}{\hbar^2} - \xi R(x)\right] \phi(x) = 0 \tag{4.2}$$

so that the Feynman Green's function must obey

$$\int dV_{x'} F(x, x') G_F(x', x'') = -\delta\left(x - x''\right) \tag{4.3}$$

Due to the symmetry of $G_F(x, x')$, the same equation must also hold at the point $x'$.

Following Schwinger, we define a set of basis vectors $|x\rangle$ of some abstract Hilbert space which obey $x^\mu |x'\rangle = x'^\mu |x'\rangle$. Then

$$\delta(x - x') = \langle x|1|x'\rangle \tag{4.4a}$$

$$G_F(x, x') = \langle x|G_F|x'\rangle \tag{4.4b}$$

$$F(x, x') = \langle x|F|x'\rangle \tag{4.4c}$$

where $G_F$ and $F$ are now operators whose matrix values give the Feynman Green's function and differential operator, respectively. Note that the operator $F$, defined above, transforms as a tensor density of weight one (due to the leading factor of $g^{1/2}$) so that $g^{-1/4}Fg^{-1/4}$ transforms as a tensor. With this in mind it is advantageous to write

$$-1 = FG_F = g^{-1/4}Fg^{-1/4}g^{1/4}Gg^{1/4} \tag{4.5}$$

so that

$$g^{1/4}G_Fg^{1/4} = -\frac{1}{g^{-1/4}Fg^{-1/4}+i\epsilon} = i\int_0^\infty ds\; e^{i(g^{-1/4}Fg^{-1/4}-i\epsilon)s}.$$

where the small positive imaginary part is in order to select the correct contour of integration needed to obtain the desired propagator. Sandwiching this expression between the appropriate eigenkets to determine the position-space representation of these operators then gives (dropping for now the small imaginary parts)

$$G_F(x,x') = i\int_0^\infty ds\; g^{-1/4}(x)\left\langle x\left|e^{i\,g^{-1/4}Fg^{-1/4}\,s}\right|x'\right\rangle g^{-1/4}(x'). \tag{4.6}$$

This motivates the defition of the "transition amplitude"

$$\left\langle x,s\right|x',0\left.\right\rangle \equiv \left\langle x\left|e^{i\,g^{-1/4}Fg^{-1/4}\,s}\right|x'\right\rangle \tag{4.7}$$

with the operator $F$ playing the role of the Hamiltonian and $s$ playing the role of time (which motivates the definition of $s$ as the "proper time"). Note that by construction this amplitude satisfies a 'Schrodinger' equation

$$-\frac{\partial}{\partial s}\left\langle x,s\right|x',0\left.\right\rangle = -F\left\langle x,s\right|x',0\left.\right\rangle \tag{4.8}$$

which, when compared to (3.52), makes it evident that this is just the heat kernel $K$ as previously defined. From this we define the kernel $K$ such that the Green's function is given by

$$G_F(x,x') = i\int_0^\infty ds\; g(x)^{-1/4}K(x,x';s)g(x')^{-1/4} \tag{4.9}$$

where, in the case of the theory determined by action (3.1), implies that $K(x,x';s)$ necessarily obeys

$$\left(i\frac{\partial}{\partial s}+\Box_x-\frac{m^2c^2}{\hbar^2}-\xi R(x)\right)K=0 \tag{4.10}$$

with the boundary condition

$$K(x,x';0)=\delta(x-x'). \tag{4.11}$$

Using the transition amplitude interpretation (4.7) for $K$ motivates this initial condition. By comparison to the flat-spacetime solution the following ansatz is made

$$K(x,x';s)=-\frac{i}{(4\pi)^2}\frac{D^{1/2}(x,x')}{s^2}e^{i\left(\frac{\sigma(x,x')}{2s}-\frac{m^2c^2}{\hbar^2}s\right)}\Lambda(x,x';s) \tag{4.12}$$

where $\hat{D}(x,x')$ is the Van Vleck-Morette determinant defined by

$$\hat{D}(x,x')=\left|\det\hat{D}_{\mu\nu'}\right|=\left|\det\nabla_\mu\nabla_\nu\sigma(x,x')\right| \tag{4.13}$$

and

$$\Lambda(x,x;0)=1. \tag{4.14}$$

### 4.1.1 Asymptotic expansion

Evaluating (4.10) on (4.12) yields a differential equation for $\Lambda$, a series-solution ansatz is made in the form

$$\Lambda(x,x';s)=\sum_{k=0}^{\infty}a_k(x,x')(is)^k \tag{4.15}$$

with $a_0(x,x)=1$. The differential equation for $\Lambda$ now provides recursion relations for calculating the $a_k$ functions.

It is important to note the applicability of the ansatz (4.15) which fundamentally assumes that $s$ is a small parameter. In terms of (4.7) this implies that we are studying states where $\sigma(x,x')$ is very small. In this way the Schwinger-DeWitt approximation for the heat kernel (4.12)(4.15) explores only the local structure of the geometry. It will be useful, then, for computing ultraviolet divergences however will not capture any of the global properties of the theory. In particular that implies that the result is completely independent of the quantum state that the field is assumed to be in. Since particle creation is intrinsically tied to the choice of a particular quantum state, this approximation to $K$ *cannot* be used to answer our ultimate question: what is the energy momentum tensor describing the quantum field. In order to answer this it will be necessary to use a different approach for determining the kernel $K$, or something altogether different which does not utilize heat kernel at all to determine $G^{(1)}(x, x')$ and, ultimately, $\langle \tau_{\mu\nu} \rangle$.

Putting all the above together then gives

$$\begin{aligned} G(x,x') &= -\frac{1}{(4\pi)^2}\Delta^{1/2}\int_0^\infty ds\ \frac{1}{(is)^2}\left[\sum_{k=0}^{\infty} a_k\ (is)^k\right] e^{i\left(\frac{\sigma}{2s}-\frac{m^2c^2}{\hbar^2}s\right)} \\ &= -\frac{1}{(4\pi)^2}\Delta^{1/2}\left[\sum_{k=0}^{\infty} a_k\ \left(-\frac{\hbar^2}{c^2}\frac{\partial}{\partial m^2}\right)^k\right]\int_0^\infty ds\ \frac{1}{(is)^2}e^{i\left(\frac{\sigma}{2s}-\frac{m^2c^2}{\hbar^2}s\right)} \end{aligned} \tag{4.16}$$

where the integral in the last expression can be evaluated in terms of Hankel functions. The result is an expansion in inverse powers of the field-mass $m$, and as such is generally not convergent.

## 4.2 Alternate methods for computing the heat kernel

The Schwinger-DeWitt approach, leading to equation (4.12), provides an attractive inroads for calculating the Feynman Green's function $G_F$. In particular, since ultimately it is the coincidence limit of $G$ that comes in to play, it becomes useful to calculate the trace of the kernel

$$\mathrm{tr}K(s,x,x') = \int dV_x K(s,x,x). \tag{4.17}$$

Most approaches focus on trying to determine the unknown function $\Lambda$ by plugging (4.12) in to the 'Schrodinger' equation (4.10). This results in an equation for $\Lambda$

$$i\frac{\partial\Lambda}{\partial s} + \frac{i}{s}\nabla^{\mu}\Lambda_{\mu} + \triangle^{-1/2}\Box\left(\triangle^{1/2}\Lambda\right) - \xi R\Lambda = 0 \tag{4.18}$$

where $\triangle = g^{-1/2}(x)\hat{D}(x,x')g^{-1/2}(x')$ which transforms as a scalar at each point (as opposed to $\hat{D}$ which is a scalar density of unit weight at each point).

### 4.2.1 Parker's path integral approach

This approach by Leonard Parker and Jacob Bekenstein [40, 10] is unique in that it focuses purely on finding an expression for $K(s,x,x')$ rather than making an ansatz and reducing the unknowns to some new function $\Lambda$. Their approach centers on the interpretation of $K$ as the amplitude, $\langle x,s|\, x',0\rangle$, for a particle coupled to curvature to propagate from $x$ to $x'$ in some *fictitious* proper-time interval $s$. In their work they factor out the factor of $m^2$ in the exponential to write the Green's function as

$$G_F(x,x') = i\int_0^{\infty} \left\langle x,s|\, x',0\right\rangle e^{-im^2 s} ds \tag{4.19}$$

where a small negative imaginary part is added to $m^2$ to recover the proper Feynman boundary conditions. This propagator then obeys the 'Schrodinger' equation

$$i\frac{\partial}{\partial s}\left\langle x,s|\, x',0\right\rangle = (-\Box + \xi R)\left\langle x,s|\, x',0\right\rangle. \tag{4.20}$$

They then show that $\langle x, s|\, x', 0\rangle$ can be written in the path integral form

$$\langle x, s|\, x', 0\rangle = \int d[x(s')][\triangle^p] \exp\left(\frac{i}{\hbar}\int_0^s ds' \left\{\frac{1}{4}g_{\mu\nu}\frac{dx^\mu}{ds'}\frac{dx^\nu}{ds'} - \hbar^2\left[\xi + \frac{1}{3}(p-1)R(x)\right]\right\}\right), \tag{4.21}$$

where $p$ is a dimensionless parameter which is free to be chosen at will (e.g. $p = 0$ removes the Van Vleck-Morette determinant from the integral while $p = 1 - 3\xi$ removes the Ricci scalar from the exponential).

In evaluating this path integral expression the authors work in Fermi-normal coordinates relative to that particular path $x(s)$ which extremizes the integral, a coordinate system which is always locally flat at any point and with coordinate directions given by the Killing vectors at that point. In this way they can write (4.21) as an expansion, roughly, in powers of the deviation of a particular path from that one which extremizes the integral.

While this procedure furnishes, formally, a novel approximation to the heat kernel which does not depend on any asymptotic expansions in $s$, it suffers from the complexity of the resulting expression. It is not clear how the result would be applied to some generic spacetime where the geodesic structure was not already evident.

### 4.2.2 Large $s$ expansion

A substantial amount of work was done to determine an expansion for the unknown function $\Lambda$ in (4.12) but which also accounted for the large-scale contributions of quantum mechanics [3, 4, 5, 8, 9]. These approaches rely primarily on expansion in curvature scalars and thus are only well-defined for spacetimes with small and slowly-varying curvatures. While this eliminates these approaches for use here, an offshoot of their work yielded and expansion in inverse-powers of $s$ in contrast to the expansion in powers of $s$ as in (4.15) [6, 7]. Here the authors present their results based on a resummation of a perturbation series for $K$. The key result is the expression they find for the kernel at large values of $s$, thus allowing the integral e.g. in (4.9) to be evaluated for all values of the proper time. There are two chief drawbacks to this approach, one is the need to expand their work completely to curved spacetime. They have made some progress in that field however have run in to some inconsistencies. Second is the need to join their large-$s$ expansion with the small-$s$ expansion of the Schwinger-DeWitt procedure.

While the current work in this area is lacking, I believe this to be a strong candidate for future progress. It seems plausible that this could be pushed far enough to come up with an interesting result. In the following I will present a modification to the standard DeWitt-Schwinger expansion which allows for a generalization in addition to the recursion relations which correspond to the above expansion in inverse powers of $s$.

Instead of (4.12) instead consider

$$K(x,x';s) = \frac{i}{(4\pi)^2}\frac{\hat{D}^{1/2}(x,x')}{s^2}e^{i\left(\frac{\sigma(x,x')}{2s}-fs\right)}\Lambda(x,x';s) \tag{4.22}$$

where $f = f(x^\mu)$ is a function of the spacetimes coordinates. In particular it is expected that $f$ will depend on the Ricci scalar $R$ as demonstrated in [41], however the case $f = m^2$ (in dimensionless units) returns the standard calculation shown above. When inserted in to (4.10) this results in

$$\begin{aligned} 0 =& i\Lambda_{,s} + \frac{i}{s}\nabla_\mu\Lambda\nabla^\mu\sigma + \hat{D}^{-1/2}\Box\left(\hat{D}^{1/2}\Lambda\right) - \xi R\Lambda \\ &+ \left(f-m^2\right)\Lambda - is\Lambda\Box f - \Lambda\nabla_\mu\left[2is\ln\left(\hat{D}^{1/2}\Lambda\right) + s^2 f - \sigma\right]\nabla^\mu f. \end{aligned} \tag{4.23}$$

Considering momentarily only the standard asymptotic expansion (4.15) this results in the recursion relations

$$\nabla_\mu a_0 \nabla^\mu\sigma = 0 \tag{4.24a}$$

$$\begin{aligned} ka_k + \nabla_\mu a_k\nabla^\mu\sigma =& \hat{D}^{-1/2}\Box\left(\hat{D}^{1/2}a_{k-1}\right) - \xi R a_{k-1} \\ &- \Box f a_{k-2} + \nabla^\mu f\Big[a_{k-1}\nabla_\mu\sigma - 2a_{k-2}\nabla_\mu\ln\left(\hat{D}^{1/2}\right) \\ &- 2\nabla_\mu a_{k-2} + a_{k-3}\nabla_\mu f\Big] + \left(f-m^2\right)a_{k-1}. \end{aligned} \tag{4.24b}$$

Thus it is seen that the choice $f = m^2$ leads to significant simplifications. In particular it should be noted that $f \neq$ constant results in a connection between $a_k$ and $a_{k-3}$ as well as $a_{k-2}$ (note that for calculation of $a_1$ and $a_2$ it is necessary to take $a_{-2}, a_{-1} = 0$). The nature of the recurrance relations is substantially altered.

With this in mind I now consider only the case $f = m^2$ when pursuing calculations involving inverse powers of $s$. Instead of (4.15) now take the so-called "late-time" asymptotic expansion

$$\Lambda = \sum_{k=0}^{\infty} b_k (is)^{-k}. \tag{4.25}$$

The relevant recursion relations are now given by

$$kb_k - \nabla_\mu b_k \nabla^\mu \sigma = -\hat{D}^{-1/2} \Box \left( \hat{D}^{1/2} b_{k+1} \right) + \xi R b_{k+1}. \tag{4.26}$$

Thus each of the differential equations defining the new coefficients are flipped relative to their dependance on $k$ and the first equation, $k = 0$, is no longer so simple.

The solutions of this series of equations is not obvious, however (as in the case of standard DeWitt-Schwinger) it is the coincident limits of the functions $a_k$ and $b_k$ that matter. This should provide for enough simplification that the late-time asymptotics can be analyzed similarly to the "early-times" expansion.

# Chapter 5: The Energy Momentum Tensor for a Free Field in Curved Spacetime

As discussed in [48] the quantum field operator $\phi$ is not, strictly-speaking, defined at a point in spacetime as is its classical counterpart. It is formally defined only in the distributional sense, roughly speaking that is that its only well defined when integrated over spacetime against some test function. This is fine for computing observables which are linear in $\phi$, however products of fields are defined only formally and therefore it is not surprising that any naive calculation of operators such as the Green's function or the energy momentum tensor (both of which are quadratic in $\phi$) lead to divergent expectation values. They must be normalized in some manner.

When working in Minkowski space this causes no ambiguity since divergences can be resolved by use of the normal ordering. In curved spacetime however, as discussed previously, there is often no well-defined global notion of time and therefore the procedure doesn't make sense. In Minkowski space this normal ordering is equivalent to selecting the vacuum state to correspond to zero energy. Considering that particle number is an observer-dependent quantity in curved spacetime, it is not obvious what subtractions should be performed in order to bring "the" vacuum state to zero energy since different observers would disagree.

Given these ambiguities, what then are the properties of a physically realistic energy momentum tensor describing the backreaction of a quantum field on the geometry? If it is to satisfy the semi-classical Einstein equation (3.13) it should be covariantly conserved. This is true at least if it is the only source of spacetime curvature, the presence of any classical matter (such as in the interior of a star) obfuscates this fact since the Einstein equations would only imply that the *total* energy momentum tensor is conserved. It is sufficient to assume that both contributions to the total tensor, classical and quantum, are conserved individually. This might be too strong an assumption, however it greatly simplifies the mathematics.

The first section of this chapter will discuss some general properties of the mathematics behind calculations quadratic in fields and, in particular, of the energy momentum tensor. The pitfalls of straightforward attempts at calculations will be elucidated and some attempts at rectifying them will be discussed, in particular some methods suggested for calculations will be provided.

In order to answer the question posed above about properties of energy momentum tensors which are to satisfy the semi-classical Einsteint equations, Robert Wald posed a series of five axioms which should be imposed. These will be presented in the second section along with a discussion of their implications as well as their feasibility.

One method for regularization of quantities quadratic in fields, the point-splitting procedure, will be outlined in section three. While the precise details of constructions in this framework are quite technical, discussion here will be largely related to a conceptual overview of the topic. In particular it will be shown how this procedure leads to a calculation of the $a_i$ recursion coefficients from the heat kernel approach discussed previously. One major consequence of all regularization and renormalization procedures is the anomalous appearance of a trace in the energy momentum tensor for a conformally-invariant, massless scalar field. Classical theory demands that the energy momentum tensor be traceless. A quick discussion of this trace anomaly is provided as well as a discussion of its relevance to current works and connection to Hawking radiation in general.

Finally, a physical model is described which brings together many of the ideas posed so far in this work. The purpose of this section is to provide guidance when attempting a full solution to the backreaction problem, once the difficulty of calculating the energy momentum tensor is overcome. The goal of this discussion is to combine the ideas posed elsewhere in this work, as well as various other locations in the literature, in to a cohesive strategy for approaching the backreaction problem. Here I provide a novel approach, more general than all known previous works, which is meant to furnish a completely self-consistent solution to the problem.

## 5.1 General concerns

It is tempting to try and compute directly the expectation value of the energy-momentum tensor operator $\hat{\tau}[\phi]$ as given by (3.15) and (3.4), it is more useful however to re-express this as

$$\left\langle \tau^{\mu\nu}_{(\phi)} \right\rangle = \lim_{x \to x'} \hat{\tau} \left\langle \phi(x)\phi(x') + \phi(x')\phi(x) \right\rangle \tag{5.1}$$

with $\hat{\tau}$ now a differential operator. The expectation value in (5.1) is recognized as the Hadamard Green's function $G(x, x')$ which encodes information about the geometry and about the quantum state in question. Thus the problem of computing the energy-momentum tensor of a scalar field is reduced to that of finding an appropriate expression for the relevant Green's function.

For the case of a conformally invariant scalar field theory, which is given by (3.1) with the choices $m = 0$ and $\xi = 1/6$, (5.1) can be written explicitly as

$$\begin{aligned} \hat{\tau} = 2 \left( \delta_\nu{}^{\nu'} \nabla_\mu \nabla_{\nu'} + \delta_\mu{}^{\mu'} \nabla_{\mu'} \nabla_\nu \right) - g_{\mu\nu} \delta_\sigma{}^{\sigma'} \nabla_{\sigma'} \nabla^\sigma - \nabla_\mu \nabla_\nu + \\ + \delta_\mu{}^{\mu'} \delta_\nu{}^{\nu'} \nabla_{\mu'} \nabla_{\nu'} + g_{\mu\nu} \left( \nabla_\sigma \nabla^\sigma + \nabla_{\sigma'} \nabla^{\sigma'} \right) + R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \end{aligned} \tag{5.2}$$

where the primed indices indicate differentiation at the point $x'$ and the un-primed indices refer to the point $x$. Then $\delta_\mu{}^{\mu'}$ is the bitensor of parallel displacement which will be defined in section 3 but, loosely speaking, translates vector quantities from the tangent space at point $x'$ to those at the point $x$.

In our model the Green's function is implicitly a functional of $\psi$, $\lambda$, and $R$ via (2.7). A complete expression for $\langle \tau \rangle$ in terms of these un-determined functions would then furnish the energy-momentum tensor for the entire class of spherically symmetric spacetimes. With this in hand it would then be straightforward, in theory, to construct a particular collapsing-star model such as that described in (3.12) and (5.9) then go on to solve the Einstein equations for $\psi$, $\lambda$, and $R$. Disregarding the extreme difficulty in solving such non-linear equations for a general source, an expression for $\langle \tau \rangle$ in any generic class of spacetimes also lies out of reach.

While (5.1) furnishes a formal expression for the desired energy-momentum tensor, care must be taken to avoid the divergence as the limit $x \to x'$ is taken. Multiple procedures have proven effective in isolating the divergences so that they can be properly excised. Chief among them are the so-called "dimensional-regularization" and "point-splitting" procedures. In the former the spacetime dimension $n$ is kept variable while the limit $x \to x'$ is taken. The result is then expanded around $n = 4$ in order to isolate divergent terms.

The point splitting procedure involves keeping $x$ and $x'$ separate throughout the calculation while allowing $n = 4$. They points $x$ and $x'$ are considered close enough to one another that all quantities of interest can be expanded in terms of the proper-distance squared, defined as $2\sigma(x, x')$, or it's derivatives. Terms which are proportional to negative powers of $\sigma$, or to $\log(\sigma)$, will diverge as the limit $x \to x'$ is taken and $\sigma \to 0$. These are precisely the terms which must be systematically removed in order to obtain a physically-meaningful energy-momentum tensor.

The benefit of point-splitting lies in its ability to maintain covariance throughout. As such it has been used successfully to isolate the divergences in the Green's function $G(x, x')$ for a completely *arbitrary* spacetime. This result is based on the Schwinger-DeWitt expansion of the Green's function and provides, as a corollary, an enticing asymptotic expansion for the remaining, finite part which in turn can be used to construct the renormalized energy-momentum tensor. The applicability of this expansion is examined below.

One approach, first given in [2] and then in [49], relies on the conjecture that $G$ has a singularity structure of the form

$$S(x, x') = \frac{2}{(4\pi)^2} \triangle^{1/2} \left( \frac{2}{\sigma} + v \ln \sigma + w \right), \tag{5.3}$$

at least as $x \to x'$. This conjecture correctly captures the singularity structure of the Green's function in Minkowksi space and can be derived as a consequence of the heat kernel approach in curved spacetime when $m \neq 0$. The functions $\triangle^{1/2}$, $v$, and $w$ are smooth functions of both $x$ and $x'$. Hadamard showed [25] that $u$ and $v$ could be expressed as an expansion in powers of $\sigma$

$$v = \sum_{n=0}^{\infty} v_n(x, x') \sigma^n \tag{5.4a}$$

$$u = \sum_{n=0}^{\infty} u_n(x, x') \sigma^n. \tag{5.4b}$$

The function $S$ is then found by solving recursion relations for the $w_n$ and $v_n$ and an equation constraining $\triangle$ [15, 22]. This procedure uniquely specifies $S$ up to the choice of $w_0$. The choice

$w_0 = 0$ corresponds to the DeWitt-Schwinger approximation for the heat kernel (4.15) and is purely determined by the local geometry.

This ansatz for the singularity structure of $G$ may then be subtracted from the naive, straightforward (5.1) allowing the divergent parts to cancel and assuming that what remains is the physical energy momentum tensor, a procedure described in [48]. It was then pointed out that this leads to an ambiguity as it is always possible to add to $S$ a term proportional to $ln(\ell^2)$ which effectively makes the change $\sigma \to \ell^2\sigma$. This changes the fundamental length scale of the problem (since $\sigma$ measures the geodesic distance between points). For massive theories there is a preferred length scale, the Compton wavelength of the field in question. There is no such choice available in massless theories, such as the conformal theory in (5.2), and thus the ambiguity remains. This results in an ambiguity of $T_{\mu\nu}$ up to a constant multiple of a tensor constructed out of the Ricci tensor $R_{\alpha\beta}$, Ricci scalar $R$, and second derivatives of $R$.

There is also the question of the state in which the expectation value should be taken. As discussed previously, the choice of a state is tied intrinsically to a choice of observer (the one who claims that state to be "the" vacuum) and thus is not uniquely specified. Historically three choices have dominated the literature, the Unruh vacuum [46], the Hartle Hawking vacuum [26], and the Boulware vacuum [14]. All of these choices are, strictly speaking, defined only for a static Schwarzschild spacetime, however analogs in various similar geometries should be clear.

- The Unruh state is characterized by a choice of mode solutions on $\mathscr{I}^-$ which are positive frequency with respect to the Schwarzschild time coordinate $t$ and those from the past horizon to be positive frequency with respect to the null coordinate $u = t - r^*$. Physically, this corresponds to a vacuum state as measured by an observer far from the black hole and long in the past. A similar observer, far from the black hole and in the future, would then measure a flux of particles. More technically, this corresponds to a breakdown in the purely-positive frequency modes on $\mathscr{I}^+$.

- This vacuum state is characterized by the choice of mode solutions which are positive frequency with respect to the null coordinate $v = t+r^*$ on $\mathscr{I}^-$ and with respect to $u = t-r^*$ on $\mathscr{I}^+$. This results in a state which is well behaved near the black hole horizon, however any inertial observers would detect a uniform 'gas' of particles near infinity with respect to which the black hole is in equilibrium.

- The Boulware state corresponds to a choice of modes which are positive frequency with respect to $t$ everywhere. This leads to an absence of particles near infinity but with diverging energies measured by free-falling observers near the horizon.

Of the three, the Unruh vacuum state results in an energy momentum tensor which most closely resembles what is thought to be the actual physical result of a collapsing star forming a black hole.

## 5.2 Wald's axioms

While the precise procedure of regularizing and renormalizing the propagator for a field theory is rather complex, Robert Wald attempted to characterize those qualities of the resulting energy momentum tensor with the hope of specifying its form uniquely. The characteristics in his five axioms are built on the assumption that the energy momentum tensor for a quantum field should not significantly deviate from that of classical matter, nor should it deviate too far from the naive classical result (5.1).

Unless othwerwise specified, in the following $\tau_{\mu\nu}$ refers to the *renormalized*, physically-relevant, energy momentum operator and $\langle\tau_{\mu\nu}\rangle$ refers to its expectation value in a particular state. As described in his 1977 paper [48], these axioms are

1. The formal expression for $\tau_{\mu\nu}$ (non-renormalized, from (5.1)) is valid for computing the matrix element $\langle\Phi|\tau_{\mu\nu}|\Psi\rangle$ between any two orthogonal states, $\langle\Phi|\Psi\rangle = 0$.

2. Normal ordering is valid in Minkowski spacetime (with the usual $\mathbb{R}^4$ topology).

3. For any state, the expectation value $\langle\tau_{\mu\nu}\rangle$ is conserved, $\nabla^\mu\langle\tau_{\mu\nu}\rangle = 0$.

4. Causality:

   (a) For a fixed "in" state, $\langle\tau_{\mu\nu}\rangle$ at a point $p$ in spacetime depends only on the spacetime geometry to the causal past of $p$.

   (b) For a fixed "out" state, $\langle\tau_{\mu\nu}\rangle$ at a point $p$ in spacetime depends only on the spacetime geometry to the causal future of $p$.

5. For a fixed "in" or "out" state, $\langle\tau_{\mu\nu}\rangle$ varies continuously with the spacetime metric in a sufficiently strong manner to guarantee that the dynamics of the semi-classical theory, (3.13), be of the same *nature* as the dynamics of the classical theory of relativity.

The first two axioms ensure that the resulting energy momentum tensor is not wildly qualitatively different then one might get if making a similar construction in flat spacetime. The third and fourth axioms demand that the semi-classical theory reduces to the purely classical theory in the appropriate limit. The fifth axiom is the most technical and also the one which is most in doubt. It will be discussed further below. Note that none of these axioms demand any sort of local positive energy condition which is a feature of classical general relativity. Indeed it is expected that the energy density of a quantum field will in fact be negative, which is what gives rise to interesting phenomena in the semi-classical theory.

As stated by Wald, the fifth axiom is put in place to demand that the *nature* of the Einstein equations is unchanged in the semi-classical theory. That is, that it remains a second order set of differential equations. He points out that if there are terms higher order in derivatives (e.g. $\nabla_\mu \nabla_\nu R$), no matter how small their pre-factor in the set of differential equations, the characteristics of the equations is fundamentally altered. This includes changing the nature of stability properties of solutions.

In his subsequent works he points out that many procedures for computing a renormalized energy momentum tensor struggle with satisfying this axiom. He argues that the point-splitting procedure, for example, can be easily shown to obey axioms one through four but it is unclear if it always satisfies axiom five. Indeed, it has been shown that renormalizing the quantum field naturally leads to the inclusion of higher-order terms in the Einstein-Hilbert action and therefore higher-order in derivative terms in the resulting Einstein equations. Thus it is unclear if the fundamental argument he outlines supporting the need for axiom five is still sound.

Axioms three and four themselves can nearly constrain the form of the energy momentum tensor, as will be seen in the subsequent chapter. They form the backbone of the calculation outlined there.

## 5.3 Point splitting

There are multiple ways to renormalize the expectation value of (5.1). One particular method, dubbed "point splitting," benefits from remaining completely covariant throughout. The general strategy for utilizing this method is described in [12] as:

(1) Solve the field equations ( $\Box f(x^\mu) = 0$) for a complete set of normal modes from which particle states may be defined.

(2) Construct $G(x, x')$ (the Hadamard Green's function) as a mode sum.

(3) Form $G_{\text{ren}}(x, x')$ by subtracting the expression for the Green's function obtained via the Schwinger-DeWitt procedure (4.15), truncating the expansion of $G_{\text{DS}}$ at order $n$.

(4) Operate on $G_{\text{ren}}(x, x')$ (e.g. with (5.2)) to form $\langle 0 \,| \tau_{\mu\nu}(x, x') |\, 0 \rangle_{\text{ren}}$, discarding any terms of adiabatic order greater than $n$ which have appeared from differentiation of terms in ${}^{(n)}G_{\text{DS}}$.

(5) Let $x' \to x$ and display the finite result $\langle 0 \,| \tau_{\mu\nu}(x) |\, 0 \rangle_{\text{ren}}$.

Where the state $|0\rangle$ will depend on the definition of positive modes in the first step.

The divergent terms, referred to as $G_{\text{DS}}$ in the above, were worked out for a completely arbitrary spacetime in [17]. This procedure was used with some success in [16] to determine the form of the stress tensor in a Schwarzschild background geometry, however the results were limited to the surface of the horizon and to infinity

The point-splitting method as used by Christensen in [17] relies on the Schwinger-DeWitt proper time method, using an effective action to determine the form of the Green's functions through various expansions. The two points in the Green's functions are separated by a non-null geodesic in order to preserve covariance. The energy-momentum tensor can then be constructed from these two-point functions by taking the appropriate combinations of derivatives, when the two-points are brought in to coincidence the tensor diverges.

The entire utility of this procedure relies on the fact that the divergent, as well as finite but direction-dependent, terms are known to be completely independent of quantum state. That is, they depend only on the local geometry and can be constructed out of various geometric scalars. The method briefly described in the previous paragraph is tailored towards computing these state-independent terms, however in its present form it does not make any statements about the remaining, finite portion of the energy-momentum tensor (which is, of course, the ultimate goal of this work). Schematically this procedure allows for the decomposition:

$$\left\langle \tau^{\mu\nu}_{\text{quantum}} \right\rangle = \begin{pmatrix} \text{divergent} \\ \text{terms} \end{pmatrix} + \begin{pmatrix} \text{finite,} \\ \text{direction-dependent} \\ \text{terms} \end{pmatrix} + \begin{pmatrix} \text{finite,} \\ \text{direction-independent} \\ \text{terms} \end{pmatrix} . \quad (5.5)$$

where the first two sets of terms are undesirable and have been completely computed by S. Christensen in [17, 18, 19] in terms of geometric invariants. It is very clear that the divergent terms must be removed in order to obtain a physically meaningful result, the fate direction-dependent terms is not so clear. The direction-dependence that crops up is entirely determined by the choice of geodesic used to separate the points, thus if these terms are kept the resulting expression is not uniquely determined but rather depends on how the calculation is performed. While this is clearly a strong argument for the need to also subtract these terms, there remains the possibility that averaging the direction-dependence over all possible directions will result in a physically meaningful result. A possible averaging technique is outlined in [2], the result of which is

$$\begin{aligned}
\langle \tau^{\mu\nu} \rangle_{finite,avg} = -\frac{g^{1/2}}{8\pi^2}\Bigg( &- \frac{1}{4320} R_{;}{}^{\mu\nu} - \frac{1}{36}\eta R_{;}{}^{\mu\nu} + \frac{1}{1440} R^{\mu\nu\alpha}_{\ \ \ \ ;\alpha} - \frac{1}{12}\eta R^{\mu\nu\alpha}_{\ \ \ \ ;\alpha} - \frac{1}{8} m^4 g^{\mu\nu} \\
&- \frac{1}{144} m^2 R g^{\mu\nu} + \frac{1}{1152} R^2 g^{\mu\nu} - \frac{1}{8} m^2 \eta R g^{\mu\nu} + \frac{1}{48}\eta R^2 g^{\mu\nu} + \frac{5}{16}\eta^2 R^2 g^{\mu\nu} \\
&+ \frac{1}{1728} R_{;\alpha}{}^{\alpha} g^{\mu\nu} + \frac{13}{144}\eta R_{;\alpha}{}^{\alpha} g^{\mu\nu} + \frac{3}{16}\eta^2 R_{;\alpha}{}^{\alpha} g^{\mu\nu} - \frac{7}{1728} R_{\alpha\beta} R^{\alpha\beta} g^{\mu\nu} \\
&- \frac{1}{24}\eta R_{\alpha\beta} R^{\alpha\beta} g^{\mu\nu} + \frac{1}{4320} R_{\alpha}^{\ \alpha} R^{\alpha\mu} + \frac{1}{36}\eta R_{\alpha}^{\ \nu} R^{\alpha\mu} + \frac{7}{480} R_{\alpha}^{\ \mu} R^{\alpha\nu} + \frac{1}{72}\eta R_{\alpha}^{\ \mu} R^{\alpha\nu} \\
&+ \frac{5}{72} m^2 R^{\mu\nu} - \frac{1}{288} R R^{\mu\nu} - \frac{1}{4} m^2 \eta R^{\mu\nu} - \frac{1}{24}\eta R R^{\mu\nu} - \frac{1}{4}\eta^2 R R^{\mu\nu} \\
&+ \frac{1}{864} R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} g^{\mu\nu} + \frac{1}{48}\eta R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} g^{\mu\nu} - \frac{1}{1440} R_{\alpha\beta\gamma}^{\ \ \ \ \nu} R^{\alpha\beta\gamma\mu} - \frac{1}{864} R_{\alpha\beta\gamma}^{\ \ \ \ \mu} R^{\alpha\beta\gamma\nu} \\
&- \frac{1}{48}\eta R_{\alpha\beta\gamma}^{\ \ \ \ \mu} R^{\alpha\beta\gamma\nu} - \frac{1}{240} R_{\alpha\beta} R^{\alpha\nu\beta\mu} \Bigg)
\end{aligned} \quad (5.6)$$

where $\eta$ is a coupling constant, $\eta = 0$ corresponding to the conformally-coupled case and $\eta = 1/6$ to the minimally-coupled case. The individual components expressed in the coordinate system (2.7), under the condition that $R(t,r) = r$, are included in appendix A.

One avenue of exploration was the suitability of these finite terms for playing the role of the entire quantum energy-momentum tensor. However they are *not* covariantly conserved by themselves. While this might be expected on the interior where energy and momentum is constantly being exchanged back and forth from the quantum field to the classical matter, on the exterior there is no other matter field to exchange with and the tensor must be conserved. This seems to indicate that these terms, if included at all, cannot in and of themselves be the entire energy-momentum tensor.

One result that followed from pursing the point-splitting procedure to renormalize the energy momentum tensor is the "trace-anomaly." This refers to the anomalous trace given to the energy momentum tensor of a conformally invariant theory. Classically, this tensor must have zero trace. However the act of renormalizing the divergences in $G^{(1)}(x,x')$ or, what is the same, $\tau^{\mu\nu}$ results in a nonzero trace. Depending on the regularization procedure this is a result either of the imprint of the trace leftover in dimensions other that four (from dimensional regularization), or as a necessary result of subtracting terms which diverge as split points are brought back together (from point-splitting). Regardless of the procedure used the result is the same. In particular

$$T^{\alpha}_{\ \alpha} = \frac{-1}{2880\pi^2}\left(C^2 + R^{\mu\nu}R_{\mu\nu} - \frac{1}{3}R^2 + \Box R\right) \tag{5.7}$$

in four dimensions [20], where $C$ is the Weyl scalar.

One important aspect of this is the fact that, at least in the case of $m = 0$ and $\xi = 1/6$ in (3.1), one bit of information about $\tau^{\mu\nu}$ is now known. It can be shown that in four dimensions there are only four functions that need be determined to fix the value of the energy momentum tensor with one fewer degree of freedom in two dimensions, thus knowledge of this trace can then reduce the number of unknowns by one. This will be extensively in Chapter 6.

It has been shown in [20] that, in two dimensions, knowledge of the trace is synonymous with knowledge of the Hawking flux. Thus this trace anomaly is intrinsically tied to the energy flux that is being explored in this dissertation. The situation is more complicated in four dimensions, but it is clear that there is still a connection between Hawking radiation and $\tau^{\mu}_{\ \mu}$.

## 5.4 A physical model

The mathematical machinery necessary to complete a semi-classical calculation has been explored in the previous chapters. What remains is to put it to use in order to form a complete toy model of gravitational collapse to form a black hole. Conceptually, what features might one expect in such a model? It will consist of two, perhaps three, distinct regions which must be pieced together as in 2.3. It should contain both classical matter and quantum fields, both of which should source spacetime curvature. That is, it should not begin with the specification of a known exact-solution to the Einstein equations rather it must be general enough that the inclusion of each source should affect the evolution of spacetime curvature. Finally, a complete model should be able to predict the worldline describing the outer-most boundary of classical matter during collapse while also allowing energy to radiate to infinity via the Hawking mechanism.

Ideally, a completely arbitrary spacetime metric should be used to compute the (properly-renormalized) energy momentum tensor for the quantum scalar field. The sum of this tensor with the energy momentum tensor of the classical matter (such as, say, that of a perfect fluid or of a null dust) will form the total source for the Einstein field equations. These field equations would then be solved for the un-determined metric coefficients which will inevitably appear on both sides of the equation: their second derivatives defining the curvature on the right hand side while they also dictate the form of the energy momentum tensor acting as the source on the left hand side. Spherical symmetry is a reasonable approximation to make in order to simplify the form of the energy momentum tensor as well as reduce the number of un-determined functions that need to be solved for in order to fully specify the spacetime.

It can be reasonably expected that the un-determined metric functions will appear inside an integral on the 'source' side of the equation. While classical geometry is a purely local theory the expected particle creation due to quantum effects is a global phenomena. Thus the Einstein equations, for a totally general spacetime metric, will be a set of non-linear, second order, integro-differential equations for the two remaining functions necessary to describe the geometry completely (assuming spherical symmetry).

A complete model may start with two metrics defined in such a way to cover different regions of spacetime, the 'interior' and 'exterior' regions. While both are, in general, spherically symmetric it is reasonable to approximate the 'exterior' region as a radiating Vaidya metric and the 'interior' as a closed FLRW metric. The exact solution may then be though of as a pair of perturbation to each of these two regions. The exact solutions, properly joined along at some (time-dependent) value of a radial coordinate, can be used to compute the value of the quantum scalar field energy momentum tensor. This value might then be included as a source of the first-order correction to the exact-solutions used initially. In this way a perturbative solution might be constructed.

Consider a collapsing star modeled as a spherically-symmetric distribution of perfect fluid, in addition to the classical matter include a scalar a scalar field $\phi$ governed by (3.1). A spherically-symmetric spacetime can be described by the standard spherically symmetric metric tensor (2.7)

$$ds^2 = -e^{2\phi(t,r)}dt^2 + e^{2\lambda(t,r)}dr^2 + R^2(t,r)d\Omega^2. \tag{5.8}$$

The energy momentum tensor of perfect-fluid energy can then be expressed as per usual as

$$T^{\mu\nu}_{\text{(fluid)}} = \begin{cases} (\rho + P)\, u^\mu u^\nu + P g^{\mu\nu} + & r \leq r_0 \\ 0 & r > r_0. \end{cases} \tag{5.9}$$

where $P$ and $\rho$ are pressure and energy density, respectfully, as measured by an observer co-moving with the fluid, and $u^\mu(x)$ is the four-velocity of an element of fluid located at $x$, the function $r_0 = r_0(t)$ then defines the (dynamic) boundary of a collapsing ball of matter. This serves as the classical contribution to the source of gravitation in (3.11).

The second contribution to the source in (3.11), namely $\langle \tau^{\mu\nu} \rangle$, is then a functional of the three metric-functions $\phi$, $\lambda$, and $R$. This energy-momentum tensor is constructed as described in chapters 4.2.2 and 6 and must obey (3.13), thus it provides the link between quantum and classical effects.

In the region exterior to the collapsing star of our model, $r > r_0$, the only source of curvature comes from the quantum field $\phi$. Thus at large $r$ it is expected that the spacetime is well approximated by the Schwarzschild solution. At intermediate $r$ values there should be an outward flux of Hawking radiation that may cause appreciable deviations from Schwarzschild spacetime, for a massless field $\phi$ this region may be well modeled by a Vaidya-type solution [47, 32] as in [36]. Near, but exterior to, the star the expected form of the metric is unknown as both radiation and spacetime polarization should contribute to the source.

# Chapter 6: A Derivation of the Energy Momentum Tensor From Conservation Laws

A direct calculation of the energy momentum tensor for a free quantum scalar field in curved spacetime may, in principal, be computed using the methods of the previous chapter. The energy momentum tensor can be expressed as expectation values of a specified combination of derivatives of the appropriate Green's function. This calculation should be done with a point-split Green's function so that any divergent terms can be subtracted off before taking the coincidence limit. That is the basic recipe. However each step of this process comes with a host of complications, this is especially true when working in a general spherically-symmetric and fully time-dependent spacetime. Previous results typically relied on utilizing known exact solutions of Einstein's equations, specifying the spacetime background on which the calculations were being done. Those that didn't make this simplifying assumption chose instead to work with time-independent metrics and thus applied only to static models. While these approximations make the calculations more tractable, they narrow the range of applicability of their work significantly.

General relativity alone, without any reference to quantum phenomena, places constraints on the energy momentum tensor. Namely, it must be symmetric while also being covariantly conserved. Additionally, as was showed in Chapter 2, there is an additional constraint placed by the conservation of the Kodama current. These give rise to five independent equations to determine the six independent functions needed to specify any four-dimensional, rank-two tensor which respects the spherical symmetry of the geometry. This leaves only one function to be determined. The trace of the energy momentum tensor, when properly renormalized, is known exactly in the case of a massless and conformal scalar field. This is the 'trace-anomaly' calculated first in [17]. This can be used to constrain the last unknown function. Throughout this chapter the trace of the energy momentum tensor will be assumed to be known exactly and all results will be derived in terms of it.

The first section begins by calculating a formal solution for the unknown functions defining the energy momentum tensor *without* use of the Kodama vector as an additional constraint. This is a direct extension of [20] to the case of a non-static spacetime. The solution is written in terms of the un-known metric functions as well as the trace of the energy momentum tensor and the single function which defines it on a submanifold of constant radius and time. The solution is written as a pair of integrals of the unknown functions over the entire causal past of the undetermined functions. This result nearly furnishes a complete solution to the problem of determining the energy momentum tensor for a generic curved background spacetime.

The Kodama vector can then be introduced to constrain this function which defines the spherical sector of the energy momentum tensor. The statement that the Kodama current is conserved uses the assumption that the given energy momentum tensor *also* satisfies the Einstein equations. In this way the Kodama conservation equation is tied to the Einstein equations. Section two works out the consequences of this additional constraint and provides a set of equations which, when solved, should completely determine the energy momentum tensor for any geometry in the class of spherically-symmetric spacetimes. While the set of equations resist a formal, analytical solution they do represent a new approach to the problem and may allow for substantial progress to be made towards solving the larger backreaction problem.

## 6.1 Conservation laws

Here I treat the metric functions as independent quantities with the thought that they will, in principle, be determined by the Einstein equations once the form of $T^{\mu}{}_{\nu}$ is known. The trace of the tensor is considered known via other means (namely, [17]) and, as such, is considered an independent function in this analysis. This section largely follows [20], extending it to the case of a non-static spacetime.

Consider the general spherically-symmetric metric (2.7) with $R^2(t,r) = \exp(\psi(t,r))$

$$ds^2 = -e^{2\phi(t,r)}dt^2 + e^{2\lambda(t,r)}dr^2 + e^{\psi(t,r)}d\Omega^2 \tag{6.1}$$

and write down the most general energy-momentum tensor which is spherically symmetric

$$T^{\mu}{}_{\nu} = \begin{pmatrix} T^{t}{}_{t}(t,r) & T^{t}{}_{r}(t,r) & 0 & 0 \\ T^{r}{}_{t}(t,r) & T^{r}{}_{r}(t,r) & 0 & 0 \\ 0 & 0 & T^{\theta}{}_{\theta}(t,r) & 0 \\ 0 & 0 & 0 & T^{\phi}{}_{\phi}(t,r) \end{pmatrix}. \tag{6.2}$$

This metric still has some gauge freedom remaining: i.e. I could, in principle, choose to redefine the radial coordinate such that $e^{\psi} = r^2$ without any loss of generality. For now I choose to keep this form of the metric because of the ease with which special cases can be examined without the need to enact a coordinate transformation.

In order to ensure that $T^{\mu}{}_{\nu}$ is a symmetric tensor it must obey

$$T^{t}{}_{r}(t,r) = -e^{2\lambda-2\phi}T^{r}{}_{t}(t,r). \tag{6.3}$$

Further, $0 = \nabla_{\mu}T^{\mu}{}_{\theta}$ implies that

$$T^{\phi}{}_{\phi}(t,r) = T^{\theta}{}_{\theta}(t,r) \equiv T_{\Omega}(t,r). \tag{6.4}$$

The trace identity

$$T^{\alpha}{}_{\alpha} = T^{t}{}_{t} + T^{r}{}_{r} + 2T_{\Omega} \tag{6.5}$$

will be used to replace $T^t{}_t$ in the following. With all of this the remaining two non-trivial conservation equations are

$$0 = \nabla_\mu T^\mu{}_t = -\frac{\partial}{\partial t} T^r{}_r - T^r{}_r \frac{\partial}{\partial t} (2\lambda + \psi) + \frac{\partial}{\partial r} T^r{}_t + T^r{}_t \frac{\partial}{\partial r} (\phi + \lambda + \psi) - W \tag{6.6a}$$

$$0 = \nabla_\mu T^\mu{}_r = \frac{\partial}{\partial r} T^r{}_r + T^r{}_r \frac{\partial}{\partial r} (2\phi + \psi) - e^{2(\lambda - \phi)} \left[ \frac{\partial}{\partial t} T^r{}_t - T^r{}_t \frac{\partial}{\partial t} (\phi - 3\lambda - \psi) \right] + V \tag{6.6b}$$

where I have defined the in-homogeneous terms as

$$W(t,r) = 2\frac{\partial}{\partial t} T_\Omega + T_\Omega \frac{\partial}{\partial t} (2\lambda + 3\psi) - \frac{\partial}{\partial t} T^\alpha{}_\alpha - T^\alpha{}_\alpha \frac{\partial}{\partial t} (\lambda + \psi) \tag{6.7a}$$

$$V(t,r) = T_\Omega \frac{\partial}{\partial r} (2\phi - \psi) - T^\alpha{}_\alpha \frac{\partial}{\partial r} \phi. \tag{6.7b}$$

A general solution should solve (6.6) for $T^r{}_r$ and $T^r{}_t$ in terms of the remaining un-known functions.

Rather than tackle (6.6) head-on I ask a different question entirely: is there a coordinate system in which a solution is easier to come by, and if so what coordinate transformation takes me there? In principle this is just a re-shuffling of ignorance, trading in one set of pde's for another. The utility in this approach lies in the fact that the tensor transformation laws utilize the Jacobian, so a knowledge of only the derivatives may suffice rather than being obligated to actually solve the set of differential equations.

The easiest coordinate system in which the conservation laws are solvable is one in which the line element takes the form

$$ds^2 = e^{2C} \left( -d\tilde{t}^2 + d\tilde{r}^2 \right) + e^\psi d\Omega^2. \tag{6.8}$$

The problem now has two parts: (1) solve the conservation equations in this coordinate system, (2) determine the coordinate transformation which returns the metric to the originally posited form, (6.1).

### 6.1.1 Solving the conservation equations

In the new coordinate system a general, spherically symmetric energy-momentum tensor takes the form

$$T^{\mu}{}_{\nu} = \begin{pmatrix} T^{\alpha}{}_{\alpha} - T^{\tilde{r}}{}_{\tilde{r}} - 2T_{\Omega} & -T^{\tilde{r}}{}_{\tilde{t}} & 0 & 0 \\ T^{\tilde{r}}{}_{\tilde{t}} & T^{\tilde{r}}{}_{\tilde{r}} & 0 & 0 \\ 0 & 0 & T_{\Omega} & 0 \\ 0 & 0 & 0 & T_{\Omega} \end{pmatrix}. \tag{6.9}$$

I will pre-emptively define the following quantities:

$$\tilde{W} = -2\frac{\partial}{\partial\tilde{t}}T_{\Omega} - T_{\Omega}\frac{\partial}{\partial\tilde{t}}\left(2C + 3\psi\right) + \frac{\partial}{\partial\tilde{t}}T^{\alpha}{}_{\alpha} + T^{\alpha}{}_{\alpha}\frac{\partial}{\partial\tilde{t}}\left(C + \psi\right) \tag{6.10a}$$

$$\tilde{V} = T_{\Omega}\frac{\partial}{\partial\tilde{r}}\left(2C - \psi\right) - T^{\alpha}{}_{\alpha}\frac{\partial}{\partial\tilde{r}}C \tag{6.10b}$$

as well as

$$\alpha \equiv \psi + 2C \tag{6.11a}$$

$$\beta \equiv \psi - 2C. \tag{6.11b}$$

The conservation equations then become

$$0 = \nabla_{\mu}T^{\mu}{}_{\tilde{t}} = -e^{-\alpha}\frac{\partial}{\partial\tilde{t}}\left(e^{\alpha}T^{\tilde{r}}{}_{\tilde{r}}\right) + e^{-\alpha}\frac{\partial}{\partial\tilde{r}}\left(e^{\alpha}T^{\tilde{r}}{}_{\tilde{t}}\right) - \tilde{W} \tag{6.12a}$$

$$0 = \nabla_{\mu}T^{\mu}{}_{\tilde{r}} = e^{-\alpha}\frac{\partial}{\partial\tilde{r}}\left(e^{\alpha}T^{\tilde{r}}{}_{\tilde{r}}\right) - e^{-\alpha}\frac{\partial}{\partial\tilde{t}}\left(e^{\alpha}T^{\tilde{r}}{}_{\tilde{t}}\right) + \tilde{V}. \tag{6.12b}$$

Linear combinations of the derivatives of these equations will de-couple them, i.e.:

$$0 = \frac{\partial}{\partial\tilde{r}}\left[e^{\alpha}\cdot(6.12a)\right] + \frac{\partial}{\partial\tilde{t}}\left[e^{\alpha}\cdot(6.12b)\right] \tag{6.13a}$$

$$0 = \frac{\partial}{\partial\tilde{t}}\left[e^{\alpha}\cdot(6.12a)\right] + \frac{\partial}{\partial\tilde{r}}\left[e^{\alpha}\cdot(6.12b)\right] \tag{6.13b}$$

lead to the equations

$$0 = \frac{\partial^2}{\partial\tilde{r}^2}\left(e^{\alpha}T^{\tilde{r}}{}_{\tilde{r}}\right) - \frac{\partial^2}{\partial\tilde{t}^2}\left(e^{\alpha}T^{\tilde{r}}{}_{\tilde{r}}\right) + \frac{\partial}{\partial\tilde{r}}\left(e^{\alpha}\tilde{V}\right) - \frac{\partial}{\partial\tilde{t}}\left(e^{\alpha}\tilde{W}\right) \tag{6.14a}$$

$$0 = \frac{\partial^2}{\partial \tilde{r}^2}\left(e^{\alpha} T^{\tilde{r}}{}_{\tilde{t}}\right) - \frac{\partial^2}{\partial \tilde{t}^2}\left(e^{\alpha} T^{\tilde{r}}{}_{\tilde{t}}\right) + \frac{\partial}{\partial \tilde{t}}\left(e^{\alpha}\tilde{V}\right) - \frac{\partial}{\partial \tilde{r}}\left(e^{\alpha}\tilde{W}\right). \tag{6.14b}$$

These are, of course, just inhomogeneous wave equations and therefore the solutions must be

$$T^{\tilde{r}}{}_{\tilde{r}} = e^{-\alpha}\left[f_1(\tilde{t}+\tilde{r}) + f_2(\tilde{t}-\tilde{r}) - F\right] \tag{6.15a}$$

$$T^{\tilde{r}}{}_{\tilde{t}} = e^{-\alpha}\left[f_3(\tilde{t}+\tilde{r}) + f_4(\tilde{t}-\tilde{r}) - G\right] \tag{6.15b}$$

where the $f_i$ are functions determined by the boundary conditions and

$$F = \frac{1}{2}\int_{J^-}\left[\frac{\partial}{\partial \tilde{r}}\left(e^{\alpha}\tilde{V}\right) - \frac{\partial}{\partial \tilde{t}}\left(e^{\alpha}\tilde{W}\right)\right] \tag{6.16a}$$

$$G = \frac{1}{2}\int_{J^-}\left[\frac{\partial}{\partial \tilde{t}}\left(e^{\alpha}\tilde{V}\right) - \frac{\partial}{\partial \tilde{r}}\left(e^{\alpha}\tilde{W}\right)\right] \tag{6.16b}$$

are source terms coming from the inhomogeneous part of (6.14). The integrals are over the entire causal past of the point at which the solution is evaluated.

### 6.1.2 The coordinate transformation

A transformation $t \to t(\tilde{t},\tilde{r}), \quad r \to r(\tilde{t},\tilde{r})$ which obeys

$$\frac{\partial r}{\partial \tilde{t}} = e^{\phi-\lambda}\frac{\partial t}{\partial \tilde{r}} \tag{6.17a}$$

$$\frac{\partial r}{\partial \tilde{r}} = e^{\phi-\lambda}\frac{\partial t}{\partial \tilde{r}} \tag{6.17b}$$

$$\frac{\partial}{\partial \tilde{t}}\left(e^{\phi-\lambda}\frac{\partial t}{\partial \tilde{t}}\right) = \frac{\partial}{\partial \tilde{r}}\left(e^{\phi-\lambda}\frac{\partial t}{\partial \tilde{r}}\right) \tag{6.17c}$$

$$\left(\frac{\partial t}{\partial \tilde{t}}\right)^2 - \left(\frac{\partial t}{\partial \tilde{r}}\right)^2 = e^{2C-2\phi} \tag{6.17d}$$

will take the line element (6.1) to the line element (6.8). Of the above, (6.17d) should be interpreted as the definition of the metric scale factor, $C$, while (6.17a) and (6.17b) will specify how the radial coordinate transforms once a new time coordinate is established. With this in mind the only equation which need be investigated is (6.17c). Reducing it to a first-order equation would be sufficient to obtain the tensor-transformation laws.

With the above in mind, and introducing the notation

$$\frac{\partial t}{\partial \tilde{t}} \equiv \dot{t} \quad \frac{\partial t}{\partial \tilde{r}} \equiv t', \tag{6.18a}$$

the tensor transformation law allows us to expand $T^{\mu}{}_{\nu}$ in two different coordinate bases as

$$T^{t}{}_{t} = e^{2\tilde{C}} \left[ \dot{t}^2 T^{\tilde{t}}{}_{\tilde{t}} + 2\dot{t}t' T^{\tilde{r}}{}_{\tilde{t}} - t'^2 T^{\tilde{r}}{}_{\tilde{r}} \right] \tag{6.19a}$$

$$T^{r}{}_{t} = e^{\lambda-\phi} e^{2\tilde{C}} \left[ \dot{t}t' T^{\tilde{t}}{}_{\tilde{t}} + \left( \dot{t}^2 + t'^2 \right) T^{\tilde{r}}{}_{\tilde{t}} - \dot{t}t' T^{\tilde{r}}{}_{\tilde{r}} \right] \tag{6.19b}$$

$$T^{r}{}_{r} = e^{2\tilde{C}} \left[ -t'^2 T^{\tilde{t}}{}_{\tilde{t}} - 2\dot{t}t' T^{\tilde{r}}{}_{\tilde{t}} + \dot{t}^2 T^{\tilde{r}}{}_{\tilde{r}} \right] \tag{6.19c}$$

$$\tag{6.19d}$$

where I have defined $\tilde{C} \equiv \phi - C$.

## 6.2 Conservation laws with Kodama vector

I begin now with a symmetric energy momentum tensor in the form

$$T^{\mu}{}_{\nu} = \begin{pmatrix} T^{t}{}_{t} & -e^{2\lambda-2\phi}T^{r}{}_{t} & 0 & 0 \\ T^{r}{}_{t} & T^{r}{}_{r} & 0 & 0 \\ 0 & 0 & T_{\Omega} & 0 \\ 0 & 0 & 0 & T_{\Omega} \end{pmatrix} \tag{6.20}$$

where each entry is a function of $t$ and $r$ only. Here I once again use the areal radius function $R(t,r)$ instead of its logarithm $\psi(t,r)$ for simplicity. Due to the symmetry of the spacetime it is possible to define the Kodama vector via

$$k^{\mu} = {}^{(2)}\epsilon^{ab}\nabla_b R = \frac{1}{2}e^{\psi/2-\phi-\lambda}\left(\psi_r\partial_t + \psi_t\partial_r\right) \tag{6.21}$$

where ${}^{(2)}\epsilon^{ab}$ is the embedding of the two-dimensional Levi-Civita symbol, defined in the $t$-$r$ plane, in to the full spacetime. With this definition it is possible to define a new conserved current $J^{\mu}$ by

$$J^{\mu} = G^{\mu}{}_{\nu}J^{\nu} \tag{6.22}$$

where $G^{\mu\nu}$ is the Einstein tensor. Assuming that (6.20) solves the Einstein equations (which also implies that it is covariantly conserved), the statement that (6.22) is conserved can be expressed as

$$\begin{aligned} 0 &= e^{\phi+\lambda}\, T^{\mu}{}_{\nu}\nabla_{\mu}k^{\nu} \\ &= \left(R_{,rt} - R_{,r}\lambda_{,t} - R_{,t}\phi_{,r}\right)\left(T^{r}{}_{r} - T^{t}{}_{t}\right) \\ &\quad - \left[R_{,rr} - R_{,r}\left(\phi+\lambda\right)_{,r} + e^{\lambda-2\phi}\left(R_{,tt} - R_{,t}\left(\phi+\lambda\right)_{,t}\right)\right]T^{r}{}_{t}. \end{aligned} \tag{6.23}$$

This can then be used to write $T^{r}{}_{t}$ in terms of the difference $\left(T^{r}{}_{r} - T^{t}{}_{t}\right)$ in subsequent expressions. As before, the trace of the energy momentum tensor can be used as a constraint, here one of the components of (6.20) can be eliminated in favor of the trace via

$$T^{\alpha}{}_{\alpha} = T^{t}{}_{t} + T^{r}{}_{r} + 2T_{\Omega} \tag{6.24}$$

which can be used to replace $T_\Omega$ in terms of the two unknown functions $T^{t}{}_{t}$ and $T^{r}{}_{r}$ as well as the trace $T^{\alpha}{}_{\alpha}$ which is assumed to be known.

Now the two remaining non-trivial conservation equations are

$$\nabla_\mu T^{\mu}_{t} = U_1 T^{r}{}_{r,r} + U_2 T^{r}{}_{r} + V_1 T^{t}{}_{t,r} + V_2 T^{t}{}_{t,t} + V_3 T^{t}{}_{t} - \ln(R)_t T^{\alpha}{}_{\alpha} \tag{6.25a}$$

$$\nabla_\mu T^{\mu}_{r} = F_1 T^{r}{}_{r,r} + F_2 T^{r}{}_{r,t} + F_3 T^{r}{}_{r} + G_1 T^{t}{}_{t,t} + G_2 T^{t}{}_{t} - \ln(R)_r T^{\alpha}{}_{\alpha} \tag{6.25b}$$

where the $U, V, F$ and $G$ are constructed from combinations of the metric functions $\phi$, $\lambda$, and $R$ plus their derivatives. In the special case $R(t,r) = r$, which again can be assumed without loss of generality due to the remaining gauge freedom within our metric (spherically symmetric metrics are determined entirely in terms of two unknown functions), these functions are given by

$$G_1 = -F_2 = e^{2\lambda-2\psi}\frac{\lambda_t}{\psi_r+\lambda_r} \tag{6.26a}$$

$$F_3 = -G_2 = e^{2\lambda-2\psi}\frac{\lambda_t\left(\lambda_{r,t}+\phi_{r,t}+\left(\phi_t-3\lambda_t\right)\left(\phi_r+\lambda_r\right)\right)-\left(\phi_r+\lambda_r\right)\lambda_{t,t}}{\left(\psi_r+\lambda_r\right)^2}+\phi_r+\frac{3}{r} \tag{6.26b}$$

and

$$U_1 = -V_1 = \frac{\lambda_t}{\psi_r+\lambda_r} \tag{6.27a}$$

$$U_2 = -V_3 = \frac{\lambda_{r,t}}{\phi_r+\lambda_r} - \lambda_t\left(\frac{\phi_{r,r}+\lambda_{r,r}}{\left(\psi_r+\lambda_r\right)^2} - \frac{2}{r}\frac{1}{\psi_r+\lambda_r}\right). \tag{6.27b}$$

# Chapter 7: Conclusion

Any energy momentum tensor which is to satisfy the Einstein equations must be covariantly conserved. This requirement alone is enough to express the spherically-symmetric energy momentum tensor purely in terms of two quantities, here selected to be the trace as well as the term $T_\Omega$ which defines the tensor on submanifolds of constant time and radius. This is a generalization of the work in [20] and reduces to the same result in the particular case of a static spacetime. Additionally, the requirement that the Kodama current be conserved provides an additional constraint which allows $T^\mu_{\ \nu}$ to be determined in terms of only a single remaining function, here chosen to be the trace. The cost of this additional constraint being an added complexity which makes the resulting equations less amenable to being solved formally.

In the previous chapter it has been assumed that the total energy momentum tensor is given entirely by that of the quantum scalar field. This is true, for example, outside of a collapsing star where there is no classical matter present. However in the interior region the total energy momentum tensor is a sum of two individual tensors, that of the classical matter and that of the scalar field. In principle it is the sum of the two which must be conserved according to Einstein's equations, not each one individually. Perhaps the most physically realistic model will allow the two gravitating sources, classical matter and quantum fields, to directly exchange energy and momentum by demanding that the sum of the two tensors be conserved and not the two parts. For the sake of a model which has any hope of being solved, however, it seems plausible to assume that the two pieces of matter have energy momentum tensors which are conserved individually. This might be thought of as equivalent to the statement that the two matter sources can exchange energy and momentum only by using the curvature of geometry as an intermediary.

Alternative attempts to calculate the appropriate Green's function, and ultimately the energy momentum tensor, directly from quantum principles are perhaps more attractive as they provide insight in to the underlying physics. However they are mired in layers of abstraction that, so far, resist attempts to provide any results for general, spherically symmetric spacetimes.

Ultimately it will be necessary to have some knowledge of $\tau^{\mu\nu}$ at least in a family of spacetimes to which should belong the fully self-consistent solution of the semi-classical Einstein equations. Without this piece of information the equations cannot be solved in a straightforward manner. It may, however, be enough to take a perturbative approach wherein the energy momentum tensor is found on a fixed, known background spacetime. This ‘zeroth-order’ $\tau^{\mu\nu}$ could then be used to find a ‘first-order’ correction term to the initial background spacetime, this correction then giving rise to a correction to the initial energy-momentum tensor, and so on. While this method has some promise of being useful, it has not been explored in this work and it is not clear how such model might acquire any time dependence.

It may, ultimately, require a full quantum treatment of gravity to furnish any concrete results in this field. Nearly five decades have passed since the onset of investigations of phenomena related to Hawking radiation and yet only a limited number of questions have been given satisfying answers.

## APPENDIX: AVERAGED DIRECTION-DEPENDENT FINITE TERMS

$$
\begin{aligned}
T_t{}^t = {} & -\frac{e^{-4\varphi}\left(\frac{\partial\lambda}{\partial t}\right)^4}{11520} + \frac{e^{-4\varphi}\frac{\partial\varphi}{\partial t}\left(\frac{\partial\lambda}{\partial t}\right)^3}{8640} - \frac{e^{-\lambda-2\varphi}\left(\frac{\partial\varphi}{\partial r}\right)^2\left(\frac{\partial\lambda}{\partial t}\right)^2}{8640} + \frac{1}{864}e^{-4\varphi}\left(\frac{\partial\varphi}{\partial t}\right)^2\left(\frac{\partial\lambda}{\partial t}\right)^2 \\
& - \frac{e^{-4\varphi}\frac{\partial^2\lambda}{\partial t^2}\left(\frac{\partial\lambda}{\partial t}\right)^2}{8640} - \frac{e^{-\lambda-2\varphi}\frac{\partial\lambda}{\partial r}\frac{\partial\varphi}{\partial r}\left(\frac{\partial\lambda}{\partial t}\right)^2}{2160} + \frac{e^{-\lambda-2\varphi}\frac{\partial^2\varphi}{\partial r^2}\left(\frac{\partial\lambda}{\partial t}\right)^2}{1080} - \frac{e^{-4\varphi}\frac{\partial^2\varphi}{\partial t^2}\left(\frac{\partial\lambda}{\partial t}\right)^2}{1728} \\
& + \frac{73e^{-\lambda-2\varphi}\frac{\partial\lambda}{\partial r}\left(\frac{\partial\lambda}{\partial t}\right)^2}{34560r} + \frac{101e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial r}\left(\frac{\partial\lambda}{\partial t}\right)^2}{17280r} + \frac{7e^{-\lambda-2\varphi}\left(\frac{\partial\lambda}{\partial t}\right)^2}{1152r^2} - \frac{1}{480}e^{-4\varphi}\left(\frac{\partial\varphi}{\partial t}\right)^3\frac{\partial\lambda}{\partial t} \\
& + \frac{e^{-4\varphi}\frac{\partial^3\lambda}{\partial t^3}\frac{\partial\lambda}{\partial t}}{1728} - \frac{e^{-\lambda-2\varphi}\left(\frac{\partial\varphi}{\partial r}\right)^2\frac{\partial\varphi}{\partial t}\frac{\partial\lambda}{\partial t}}{1080} - \frac{1}{576}e^{-4\varphi}\frac{\partial^2\lambda}{\partial t^2}\frac{\partial\varphi}{\partial t}\frac{\partial\lambda}{\partial t} + \frac{13e^{-\lambda-2\varphi}\frac{\partial\lambda}{\partial r}\frac{\partial\varphi}{\partial r}\frac{\partial\varphi}{\partial t}\frac{\partial\lambda}{\partial t}}{8640} \\
& - \frac{13e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial t}\frac{\partial^2\varphi}{\partial r^2}\frac{\partial\lambda}{\partial t}}{4320} + \frac{7e^{-4\varphi}\frac{\partial\varphi}{\partial t}\frac{\partial^2\varphi}{\partial t^2}\frac{\partial\lambda}{\partial t}}{2880} - \frac{e^{-4\varphi}\frac{\partial^3\varphi}{\partial t^3}\frac{\partial\lambda}{\partial t}}{2880} + \frac{e^{-\lambda-2\varphi}\frac{\partial\lambda}{\partial r}\frac{\partial^2\lambda}{\partial r\,\partial t}\frac{\partial\lambda}{\partial t}}{4320} \\
& + \frac{1}{960}e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial r}\frac{\partial^2\lambda}{\partial r\,\partial t}\frac{\partial\lambda}{\partial t} - \frac{e^{-\lambda-2\varphi}\frac{\partial^3\lambda}{\partial r^2\,\partial t}\frac{\partial\lambda}{\partial t}}{2160} - \frac{e^{-\lambda-2\varphi}\frac{\partial\lambda}{\partial r}\frac{\partial^2\varphi}{\partial r\,\partial t}\frac{\partial\lambda}{\partial t}}{1440} + \frac{e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial r}\frac{\partial^2\varphi}{\partial r\,\partial t}\frac{\partial\lambda}{\partial t}}{2880} \\
& + \frac{1}{720}e^{-\lambda-2\varphi}\frac{\partial^3\varphi}{\partial r^2\,\partial t}\frac{\partial\lambda}{\partial t} - \frac{47e^{-\lambda-2\varphi}\frac{\partial\lambda}{\partial r}\frac{\partial\varphi}{\partial t}\frac{\partial\lambda}{\partial t}}{17280r} - \frac{127e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial r}\frac{\partial\varphi}{\partial t}\frac{\partial\lambda}{\partial t}}{8640r} - \frac{41e^{-\lambda-2\varphi}\frac{\partial^2\lambda}{\partial r\,\partial t}\frac{\partial\lambda}{\partial t}}{17280r} \\
& + \frac{e^{-\lambda-2\varphi}\frac{\partial^2\varphi}{\partial r\,\partial t}\frac{\partial\lambda}{\partial t}}{288r} - \frac{e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial t}\frac{\partial\lambda}{\partial t}}{2880r^2} - \frac{1}{720}e^{-2\lambda}\left(\frac{\partial\varphi}{\partial r}\right)^4 + \frac{e^{-2\lambda}\frac{\partial\lambda}{\partial r}\left(\frac{\partial\varphi}{\partial r}\right)^3}{1440} \\
& + \frac{1}{720}e^{-2\lambda}\left(\frac{\partial\lambda}{\partial r}\right)^2\left(\frac{\partial\varphi}{\partial r}\right)^2 - \frac{11e^{-2\lambda}\frac{\partial^2\lambda}{\partial r^2}\left(\frac{\partial\varphi}{\partial r}\right)^2}{8640} + \frac{e^{-\lambda-2\varphi}\frac{\partial^2\lambda}{\partial t^2}\left(\frac{\partial\varphi}{\partial r}\right)^2}{1080} + \frac{11e^{-4\varphi}\frac{\partial^2\lambda}{\partial t^2}\left(\frac{\partial\varphi}{\partial t}\right)^2}{2880} \\
& + \frac{e^{-2\lambda}\left(\frac{\partial^2\varphi}{\partial r^2}\right)^2}{2160} - \frac{e^{-\lambda-2\varphi}\left(\frac{\partial^2\lambda}{\partial r\,\partial t}\right)^2}{2160} - \frac{1}{720}e^{-\lambda-2\varphi}\left(\frac{\partial^2\varphi}{\partial r\,\partial t}\right)^2 + \frac{e^{-4\varphi}\frac{\partial^4\lambda}{\partial t^4}}{2880} - \frac{e^{-2\lambda}\left(\frac{\partial\lambda}{\partial r}\right)^3\frac{\partial\varphi}{\partial r}}{1440} \\
& + \frac{7e^{-2\lambda}\frac{\partial\lambda}{\partial r}\frac{\partial^2\lambda}{\partial r^2}\frac{\partial\varphi}{\partial r}}{4320} - \frac{e^{-2\lambda}\frac{\partial^3\lambda}{\partial r^3}\frac{\partial\varphi}{\partial r}}{2160} - \frac{13e^{-\lambda-2\varphi}\frac{\partial\lambda}{\partial r}\frac{\partial^2\lambda}{\partial t^2}\frac{\partial\varphi}{\partial r}}{8640} - \frac{1}{480}e^{-4\varphi}\frac{\partial^3\lambda}{\partial t^3}\frac{\partial\varphi}{\partial t} \\
& + \frac{11e^{-2\lambda}\left(\frac{\partial\lambda}{\partial r}\right)^2\frac{\partial^2\varphi}{\partial r^2}}{4320} - \frac{1}{720}e^{-2\lambda}\left(\frac{\partial\varphi}{\partial r}\right)^2\frac{\partial^2\varphi}{\partial r^2} - \frac{1}{540}e^{-2\lambda}\frac{\partial^2\lambda}{\partial r^2}\frac{\partial^2\varphi}{\partial r^2} + \frac{13e^{-\lambda-2\varphi}\frac{\partial^2\lambda}{\partial t^2}\frac{\partial^2\varphi}{\partial r^2}}{4320} \\
& - \frac{37e^{-2\lambda}\frac{\partial\lambda}{\partial r}\frac{\partial\varphi}{\partial r}\frac{\partial^2\varphi}{\partial r^2}}{8640} - \frac{1}{360}e^{-2\lambda}\frac{\partial\lambda}{\partial r}\frac{\partial^3\varphi}{\partial r^3} + \frac{11e^{-2\lambda}\frac{\partial\varphi}{\partial r}\frac{\partial^3\varphi}{\partial r^3}}{4320} + \frac{e^{-2\lambda}\frac{\partial^4\varphi}{\partial r^4}}{1080} - \frac{1}{720}e^{-4\varphi}\frac{\partial^2\lambda}{\partial t^2}\frac{\partial^2\varphi}{\partial t^2} \\
& - \frac{e^{-\lambda-2\varphi}\frac{\partial\lambda}{\partial r}\frac{\partial\varphi}{\partial t}\frac{\partial^2\lambda}{\partial r\,\partial t}}{4320} - \frac{1}{540}e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial r}\frac{\partial\varphi}{\partial t}\frac{\partial^2\lambda}{\partial r\,\partial t} + \frac{e^{-\lambda-2\varphi}\frac{\partial\lambda}{\partial r}\frac{\partial^3\lambda}{\partial r\,\partial t^2}}{4320} + \frac{1}{540}e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial r}\frac{\partial^3\lambda}{\partial r\,\partial t^2} \\
& + \frac{e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial t}\frac{\partial^3\lambda}{\partial r^2\,\partial t}}{2160} - \frac{e^{-\lambda-2\varphi}\frac{\partial^4\lambda}{\partial r^2\,\partial t^2}}{2160} - \frac{e^{-\lambda-2\varphi}\frac{\partial\lambda}{\partial r}\frac{\partial\varphi}{\partial t}\frac{\partial^2\varphi}{\partial r\,\partial t}}{2880} - \frac{1}{720}e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial r}\frac{\partial^3\varphi}{\partial r\,\partial t^2} \\
& + \frac{1}{720}e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial r}\frac{\partial\varphi}{\partial t}\frac{\partial^2\varphi}{\partial r\,\partial t} + \frac{7e^{-\lambda-2\varphi}\frac{\partial^2\lambda}{\partial r\,\partial t}\frac{\partial^2\varphi}{\partial r\,\partial t}}{4320} + \frac{e^{-\lambda-2\varphi}\frac{\partial\lambda}{\partial r}\frac{\partial^3\varphi}{\partial r\,\partial t^2}}{2880} + \frac{e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial t}\frac{\partial^3\varphi}{\partial r^2\,\partial t}}{1440} \\
& - \frac{e^{-\lambda-2\varphi}\frac{\partial^4\varphi}{\partial r^2\,\partial t^2}}{1440} - \frac{e^{-2\lambda}\left(\frac{\partial\lambda}{\partial r}\right)^3}{1152r} - \frac{31e^{-2\lambda}\left(\frac{\partial\varphi}{\partial r}\right)^3}{1440r} + \frac{7e^{-2\lambda}\frac{\partial\lambda}{\partial r}\frac{\partial^2\lambda}{\partial r^2}}{3456r} - \frac{e^{-2\lambda}\frac{\partial^3\lambda}{\partial r^3}}{1728r}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{47e^{-\lambda-2\varphi}\frac{\partial\lambda}{\partial r}\frac{\partial^2\lambda}{\partial t^2}}{17280r}+\frac{41e^{-2\lambda}\left(\frac{\partial\lambda}{\partial r}\right)^2\frac{\partial\varphi}{\partial r}}{5760r}-\frac{e^{-2\lambda}\frac{\partial^2\lambda}{\partial r^2}\frac{\partial\varphi}{\partial r}}{320r}+\frac{127e^{-\lambda-2\varphi}\frac{\partial^2\lambda}{\partial t^2}\frac{\partial\varphi}{\partial r}}{8640r}-\frac{13e^{-2\lambda}\frac{\partial\lambda}{\partial r}\frac{\partial^2\varphi}{\partial r^2}}{960r}\\
&-\frac{13e^{-2\lambda}\frac{\partial\varphi}{\partial r}\frac{\partial^2\varphi}{\partial r^2}}{864r}+\frac{e^{-2\lambda}\frac{\partial^3\varphi}{\partial r^3}}{270r}+\frac{e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial t}\frac{\partial^2\lambda}{\partial r\,\partial t}}{1728r}-\frac{e^{-\lambda-2\varphi}\frac{\partial^3\lambda}{\partial r\,\partial t^2}}{1728r}+\frac{e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial t}\frac{\partial^2\varphi}{\partial r\,\partial t}}{720r}\\
&-\frac{e^{-\lambda-2\varphi}\frac{\partial^3\varphi}{\partial r\,\partial t^2}}{720r}+\frac{5e^{-2\lambda}\left(\frac{\partial\lambda}{\partial r}\right)^2}{6912r^2}-\frac{41e^{-2\lambda}\left(\frac{\partial\varphi}{\partial r}\right)^2}{2880r^2}-\frac{e^{-2\lambda}\frac{\partial^2\lambda}{\partial r^2}}{1728r^2}+\frac{e^{-\lambda-2\varphi}\frac{\partial^2\lambda}{\partial t^2}}{2880r^2}-\frac{e^{-2\lambda}\frac{\partial\lambda}{\partial r}\frac{\partial\varphi}{\partial r}}{192r^2}\\
&+\frac{e^{-2\lambda}\frac{\partial\lambda}{\partial r}3}{864r^3}+\frac{5e^{-2\lambda}\frac{\partial\varphi}{\partial r}}{432r^3}-\frac{5e^{-\lambda}\frac{\partial\varphi}{\partial r}}{432r^3}+\frac{e^{-2\lambda}}{864r^4}-\frac{e^{-\lambda}}{864r^4}
\end{aligned}
$$

$$
\begin{aligned}
T_t{}^r=&\frac{e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial r}\left(\frac{\partial\lambda}{\partial t}\right)^3}{17280}-\frac{73e^{-\lambda-2\varphi}\frac{\partial R}{\partial r}\left(\frac{\partial\lambda}{\partial t}\right)^3}{17280R}+\frac{e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial r}\frac{\partial\varphi}{\partial t}\left(\frac{\partial\lambda}{\partial t}\right)^2}{4320}-\frac{e^{-\lambda-2\varphi}\frac{\partial^2\lambda}{\partial r\,\partial t}\left(\frac{\partial\lambda}{\partial t}\right)^2}{17280}\\
&-\frac{e^{-\lambda-2\varphi}\frac{\partial^2\varphi}{\partial r\,\partial t}\left(\frac{\partial\lambda}{\partial t}\right)^2}{17280}+\frac{e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial t}\frac{\partial R}{\partial r}\left(\frac{\partial\lambda}{\partial t}\right)^2}{135R}-\frac{37e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial r}\frac{\partial R}{\partial t}\left(\frac{\partial\lambda}{\partial t}\right)^2}{4320R}+\frac{e^{-\lambda-2\varphi}\frac{\partial^2R}{\partial r\,\partial t}\left(\frac{\partial\lambda}{\partial t}\right)^2}{120R}\\
&-\frac{19e^{-\lambda-2\varphi}\frac{\partial R}{\partial r}\frac{\partial R}{\partial t}\left(\frac{\partial\lambda}{\partial t}\right)^2}{8640R^2}-\frac{e^{-2\lambda}\left(\frac{\partial\varphi}{\partial r}\right)^3\frac{\partial\lambda}{\partial t}}{4320}-\frac{e^{-2\lambda}\frac{\partial\lambda}{\partial r}\left(\frac{\partial\varphi}{\partial r}\right)^2\frac{\partial\lambda}{\partial t}}{4320}-\frac{e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial r}\left(\frac{\partial\varphi}{\partial t}\right)^2\frac{\partial\lambda}{\partial t}}{1440}\\
&+\frac{e^{-2\lambda}\left(\frac{\partial\lambda}{\partial r}\right)^2\frac{\partial\varphi}{\partial r}\frac{\partial\lambda}{\partial t}}{5760}-\frac{e^{-2\lambda}\frac{\partial^2\lambda}{\partial r^2}\frac{\partial\varphi}{\partial r}\frac{\partial\lambda}{\partial t}}{5760}-\frac{e^{-\lambda-2\varphi}\frac{\partial^2\lambda}{\partial t^2}\frac{\partial\varphi}{\partial r}\frac{\partial\lambda}{\partial t}}{4320}-\frac{e^{-2\lambda}\frac{\partial\lambda}{\partial r}\frac{\partial^2\varphi}{\partial r^2}\frac{\partial\lambda}{\partial t}}{1920}+\frac{e^{-2\lambda}\frac{\partial\varphi}{\partial r}\frac{\partial^2\varphi}{\partial r^2}\frac{\partial\lambda}{\partial t}}{2160}\\
&+\frac{e^{-2\lambda}\frac{\partial^3\varphi}{\partial r^3}\frac{\partial\lambda}{\partial t}}{2880}+\frac{e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial r}\frac{\partial^2\varphi}{\partial t^2}\frac{\partial\lambda}{\partial t}}{2880}-\frac{e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial t}\frac{\partial^2\lambda}{\partial r\,\partial t}\frac{\partial\lambda}{\partial t}}{5760}+\frac{e^{-\lambda-2\varphi}\frac{\partial^3\lambda}{\partial r\,\partial t^2}\frac{\partial\lambda}{\partial t}}{17280}+\frac{e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial t}\frac{\partial^2\varphi}{\partial r\,\partial t}\frac{\partial\lambda}{\partial t}}{2160}\\
&-\frac{e^{-\lambda-2\varphi}\frac{\partial^3\varphi}{\partial r\,\partial t^2}\frac{\partial\lambda}{\partial t}}{8640}+\frac{e^{-2\lambda}\left(\frac{\partial\lambda}{\partial r}\right)^2\frac{\partial R}{\partial r}\frac{\partial\lambda}{\partial t}}{2880R}+\frac{71e^{-2\lambda}\left(\frac{\partial\varphi}{\partial r}\right)^2\frac{\partial R}{\partial r}\frac{\partial\lambda}{\partial t}}{4320R}+\frac{e^{-\lambda-2\varphi}\left(\frac{\partial\varphi}{\partial t}\right)^2\frac{\partial R}{\partial r}\frac{\partial\lambda}{\partial t}}{1440R}\\
&-\frac{e^{-2\lambda}\frac{\partial^2\lambda}{\partial r^2}\frac{\partial R}{\partial r}\frac{\partial\lambda}{\partial t}}{2880R}-\frac{7e^{-\lambda-2\varphi}\frac{\partial^2\lambda}{\partial t^2}\frac{\partial R}{\partial r}\frac{\partial\lambda}{\partial t}}{864R}-\frac{77e^{-2\lambda}\frac{\partial\lambda}{\partial r}\frac{\partial\varphi}{\partial r}\frac{\partial R}{\partial r}\frac{\partial\lambda}{\partial t}}{8640R}+\frac{73e^{-2\lambda}\frac{\partial^2\varphi}{\partial r^2}\frac{\partial R}{\partial r}\frac{\partial\lambda}{\partial t}}{4320R}\\
&+\frac{17e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial r}\frac{\partial\varphi}{\partial t}\frac{\partial R}{\partial t}\frac{\partial\lambda}{\partial t}}{1080R}-\frac{e^{-2\lambda}\frac{\partial\lambda}{\partial r}\frac{\partial^2R}{\partial r^2}\frac{\partial\lambda}{\partial t}}{960R}+\frac{e^{-2\lambda}\frac{\partial\varphi}{\partial r}\frac{\partial^2R}{\partial r^2}\frac{\partial\lambda}{\partial t}}{1080R}+\frac{e^{-2\lambda}\frac{\partial^3R}{\partial r^3}\frac{\partial\lambda}{\partial t}}{1440R}+\frac{e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial r}\frac{\partial^2R}{\partial t^2}\frac{\partial\lambda}{\partial t}}{2160R}\\
&-\frac{e^{-\lambda-2\varphi}\frac{\partial R}{\partial t}\frac{\partial^2\lambda}{\partial r\,\partial t}\frac{\partial\lambda}{\partial t}}{8640R}-\frac{e^{-\lambda-2\varphi}\frac{\partial R}{\partial t}\frac{\partial^2\varphi}{\partial r\,\partial t}\frac{\partial\lambda}{\partial t}}{4320R}-\frac{13e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial t}\frac{\partial^2R}{\partial r\,\partial t}\frac{\partial\lambda}{\partial t}}{864R}-\frac{29e^{-2\lambda}\frac{\partial\lambda}{\partial r}\left(\frac{\partial R}{\partial r}\right)^2\frac{\partial\lambda}{\partial t}}{8640R^2}\\
&+\frac{31e^{-2\lambda}\frac{\partial\varphi}{\partial r}\left(\frac{\partial R}{\partial r}\right)^2\frac{\partial\lambda}{\partial t}}{4320R^2}-\frac{5e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial r}\left(\frac{\partial R}{\partial t}\right)^2\frac{\partial\lambda}{\partial t}}{864R^2}+\frac{e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial t}\frac{\partial R}{\partial r}\frac{\partial R}{\partial t}\frac{\partial\lambda}{\partial t}}{160R^2}+\frac{13e^{-2\lambda}\frac{\partial R}{\partial r}\frac{\partial^2R}{\partial r^2}\frac{\partial\lambda}{\partial t}}{2160R^2}\\
&-\frac{e^{-\lambda-2\varphi}\frac{\partial R}{\partial r}\frac{\partial^2R}{\partial t^2}\frac{\partial\lambda}{\partial t}}{135R^2}+\frac{23e^{-\lambda-2\varphi}\frac{\partial R}{\partial t}\frac{\partial^2R}{\partial r\,\partial t}\frac{\partial\lambda}{\partial t}}{4320R^2}-\frac{e^{-2\lambda}\left(\frac{\partial R}{\partial r}\right)^3\frac{\partial\lambda}{\partial t}}{180R^3}+\frac{e^{-\lambda-2\varphi}\frac{\partial R}{\partial r}\left(\frac{\partial R}{\partial t}\right)^2\frac{\partial\lambda}{\partial t}}{180R^3}\\
&+\frac{5e^{-\lambda}\frac{\partial R}{\partial r}\frac{\partial\lambda}{\partial t}}{864R^3}-\frac{e^{-\lambda-2\varphi}\frac{\partial^3\lambda}{\partial t^3}\frac{\partial\varphi}{\partial r}}{2880}+\frac{1}{960}e^{-\lambda-2\varphi}\frac{\partial^2\lambda}{\partial t^2}\frac{\partial\varphi}{\partial r}\frac{\partial\varphi}{\partial t}+\frac{e^{-2\lambda}\left(\frac{\partial\varphi}{\partial r}\right)^2\frac{\partial^2\lambda}{\partial r\,\partial t}}{8640}\\
&+\frac{e^{-\lambda-2\varphi}\left(\frac{\partial\varphi}{\partial t}\right)^2\frac{\partial^2\lambda}{\partial r\,\partial t}}{4320}+\frac{e^{-\lambda-2\varphi}\frac{\partial^2\lambda}{\partial t^2}\frac{\partial^2\lambda}{\partial r\,\partial t}}{8640}-\frac{e^{-2\lambda}\frac{\partial\lambda}{\partial r}\frac{\partial\varphi}{\partial r}\frac{\partial^2\lambda}{\partial r\,\partial t}}{4320}+\frac{e^{-2\lambda}\frac{\partial^2\varphi}{\partial r^2}\frac{\partial^2\lambda}{\partial r\,\partial t}}{2880}-\frac{e^{-\lambda-2\varphi}\frac{\partial^2\varphi}{\partial t^2}\frac{\partial^2\lambda}{\partial r\,\partial t}}{8640}\\
&-\frac{e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial t}\frac{\partial^3\lambda}{\partial r\,\partial t^2}}{2880}+\frac{e^{-\lambda-2\varphi}\frac{\partial^4\lambda}{\partial r\,\partial t^3}}{8640}+\frac{e^{-2\lambda}\frac{\partial\varphi}{\partial r}\frac{\partial^3\lambda}{\partial r^2\,\partial t}}{8640}-\frac{e^{-2\lambda}\left(\frac{\partial\lambda}{\partial r}\right)^2\frac{\partial^2\varphi}{\partial r\,\partial t}}{8640}+\frac{e^{-2\lambda}\left(\frac{\partial\varphi}{\partial r}\right)^2\frac{\partial^2\varphi}{\partial r\,\partial t}}{2160}
\end{aligned}
$$

$$+\frac{e^{-2\lambda}\frac{\partial^2\lambda}{\partial r^2}\frac{\partial^2\varphi}{\partial r\,\partial t}}{8640}-\frac{e^{-\lambda-2\varphi}\frac{\partial^2\lambda}{\partial t^2}\frac{\partial^2\varphi}{\partial r\,\partial t}}{2880}+\frac{e^{-2\lambda}\frac{\partial\lambda}{\partial r}\frac{\partial\varphi}{\partial r}\frac{\partial^2\varphi}{\partial r\,\partial t}}{2880}-\frac{e^{-2\lambda}\frac{\partial^2\varphi}{\partial r^2}\frac{\partial^2\varphi}{\partial r\,\partial t}}{2160}+\frac{e^{-2\lambda}\frac{\partial\lambda}{\partial r}\frac{\partial^3\varphi}{\partial r^2\,\partial t}}{2880}$$

$$-\frac{e^{-2\lambda}\frac{\partial\varphi}{\partial r}\frac{\partial^3\varphi}{\partial r^2\,\partial t}}{4320}-\frac{e^{-2\lambda}\frac{\partial^4\varphi}{\partial r^3\,\partial t}}{4320}+\frac{e^{-2\lambda}\left(\frac{\partial\varphi}{\partial r}\right)^2\frac{\partial\varphi}{\partial t}\frac{\partial R}{\partial r}}{720R}+\frac{e^{-2\lambda}\frac{\partial\lambda}{\partial r}\frac{\partial\varphi}{\partial r}\frac{\partial\varphi}{\partial t}\frac{\partial R}{\partial r}}{1440R}+\frac{19e^{-2\lambda}\left(\frac{\partial\varphi}{\partial r}\right)^3\frac{\partial R}{\partial t}}{540R}$$

$$-\frac{7e^{-2\lambda}\frac{\partial\lambda}{\partial r}\left(\frac{\partial\varphi}{\partial r}\right)^2\frac{\partial R}{\partial t}}{432R}-\frac{73e^{-\lambda-2\varphi}\frac{\partial^2\lambda}{\partial t^2}\frac{\partial\varphi}{\partial r}\frac{\partial R}{\partial t}}{4320R}+\frac{7e^{-2\lambda}\frac{\partial\varphi}{\partial r}\frac{\partial^2\varphi}{\partial r^2}\frac{\partial R}{\partial t}}{216R}+\frac{e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial r}\frac{\partial^2\varphi}{\partial t^2}\frac{\partial R}{\partial t}}{720R}$$

$$-\frac{e^{-2\lambda}\frac{\partial\varphi}{\partial r}\frac{\partial\varphi}{\partial t}\frac{\partial^2R}{\partial r^2}}{720R}+\frac{e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial r}\frac{\partial\varphi}{\partial t}\frac{\partial^2R}{\partial t^2}}{360R}-\frac{e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial r}\frac{\partial^3R}{\partial t^3}}{720R}-\frac{e^{-2\lambda}\frac{\partial\lambda}{\partial r}\frac{\partial R}{\partial r}\frac{\partial^2\lambda}{\partial r\,\partial t}}{2160R}+\frac{e^{-2\lambda}\frac{\partial\varphi}{\partial r}\frac{\partial R}{\partial r}\frac{\partial^2\lambda}{\partial r\,\partial t}}{4320R}$$

$$-\frac{e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial t}\frac{\partial R}{\partial t}\frac{\partial^2\lambda}{\partial r\,\partial t}}{2160R}+\frac{e^{-2\lambda}\frac{\partial^2R}{\partial r^2}\frac{\partial^2\lambda}{\partial r\,\partial t}}{1440R}+\frac{e^{-\lambda-2\varphi}\frac{\partial^2R}{\partial t^2}\frac{\partial^2\lambda}{\partial r\,\partial t}}{4320R}+\frac{e^{-\lambda-2\varphi}\frac{\partial R}{\partial t}\frac{\partial^3\lambda}{\partial r\,\partial t^2}}{4320R}+\frac{e^{-2\lambda}\frac{\partial R}{\partial r}\frac{\partial^3\lambda}{\partial r^2\,\partial t}}{4320R}$$

$$+\frac{e^{-2\lambda}\frac{\partial\lambda}{\partial r}\frac{\partial R}{\partial r}\frac{\partial^2\varphi}{\partial r\,\partial t}}{2160R}+\frac{e^{-2\lambda}\frac{\partial\varphi}{\partial r}\frac{\partial R}{\partial r}\frac{\partial^2\varphi}{\partial r\,\partial t}}{2160R}+\frac{e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial t}\frac{\partial R}{\partial t}\frac{\partial^2\varphi}{\partial r\,\partial t}}{540R}-\frac{e^{-2\lambda}\frac{\partial^2R}{\partial r^2}\frac{\partial^2\varphi}{\partial r\,\partial t}}{2160R}-\frac{e^{-\lambda-2\varphi}\frac{\partial^2R}{\partial t^2}\frac{\partial^2\varphi}{\partial r\,\partial t}}{720R}$$

$$-\frac{e^{-\lambda-2\varphi}\frac{\partial R}{\partial t}\frac{\partial^3\varphi}{\partial r\,\partial t^2}}{2160R}-\frac{e^{-2\lambda}\frac{\partial R}{\partial r}\frac{\partial^3\varphi}{\partial r^2\,\partial t}}{2160R}-\frac{e^{-2\lambda}\left(\frac{\partial\lambda}{\partial r}\right)^2\frac{\partial^2R}{\partial r\,\partial t}}{4320R}-\frac{5e^{-2\lambda}\left(\frac{\partial\varphi}{\partial r}\right)^2\frac{\partial^2R}{\partial r\,\partial t}}{144R}$$

$$-\frac{e^{-\lambda-2\varphi}\left(\frac{\partial\varphi}{\partial t}\right)^2\frac{\partial^2R}{\partial r\,\partial t}}{2160R}+\frac{e^{-2\lambda}\frac{\partial^2\lambda}{\partial r^2}\frac{\partial^2R}{\partial r\,\partial t}}{4320R}+\frac{71e^{-\lambda-2\varphi}\frac{\partial^2\lambda}{\partial t^2}\frac{\partial^2R}{\partial r\,\partial t}}{4320R}+\frac{71e^{-2\lambda}\frac{\partial\lambda}{\partial r}\frac{\partial\varphi}{\partial r}\frac{\partial^2R}{\partial r\,\partial t}}{4320R}$$

$$-\frac{71e^{-2\lambda}\frac{\partial^2\varphi}{\partial r^2}\frac{\partial^2R}{\partial r\,\partial t}}{2160R}-\frac{e^{-\lambda-2\varphi}\frac{\partial^2\varphi}{\partial t^2}\frac{\partial^2R}{\partial r\,\partial t}}{2160R}-\frac{e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial t}\frac{\partial^3R}{\partial r\,\partial t^2}}{720R}+\frac{e^{-\lambda-2\varphi}\frac{\partial^4R}{\partial r\,\partial t^3}}{2160R}+\frac{e^{-2\lambda}\frac{\partial\lambda}{\partial r}\frac{\partial^3R}{\partial r^2\,\partial t}}{1440R}$$

$$-\frac{e^{-2\lambda}\frac{\partial^4R}{\partial r^3\,\partial t}}{2160R}+\frac{23e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial r}\frac{\partial\varphi}{\partial t}\left(\frac{\partial R}{\partial t}\right)^2}{2160R^2}+\frac{e^{-2\lambda}\left(\frac{\partial\lambda}{\partial r}\right)^2\frac{\partial R}{\partial r}\frac{\partial R}{\partial t}}{4320R^2}+\frac{5e^{-2\lambda}\left(\frac{\partial\varphi}{\partial r}\right)^2\frac{\partial R}{\partial r}\frac{\partial R}{\partial t}}{432R^2}$$

$$-\frac{e^{-\lambda-2\varphi}\left(\frac{\partial\varphi}{\partial t}\right)^2\frac{\partial R}{\partial r}\frac{\partial R}{\partial t}}{1080R^2}-\frac{e^{-2\lambda}\frac{\partial^2\lambda}{\partial r^2}\frac{\partial R}{\partial r}\frac{\partial R}{\partial t}}{4320R^2}+\frac{e^{-\lambda-2\varphi}\frac{\partial^2\lambda}{\partial t^2}\frac{\partial R}{\partial r}\frac{\partial R}{\partial t}}{2160R^2}-\frac{11e^{-2\lambda}\frac{\partial\lambda}{\partial r}\frac{\partial\varphi}{\partial r}\frac{\partial R}{\partial r}\frac{\partial R}{\partial t}}{1440R^2}$$

$$-\frac{e^{-2\lambda}\frac{\partial^2\varphi}{\partial r^2}\frac{\partial R}{\partial r}\frac{\partial R}{\partial t}}{1080R^2}+\frac{e^{-\lambda-2\varphi}\frac{\partial^2\varphi}{\partial t^2}\frac{\partial R}{\partial r}\frac{\partial R}{\partial t}}{2160R^2}-\frac{e^{-2\lambda}\frac{\partial\lambda}{\partial r}\frac{\partial R}{\partial t}\frac{\partial^2R}{\partial r^2}}{1440R^2}+\frac{2e^{-2\lambda}\frac{\partial\varphi}{\partial r}\frac{\partial R}{\partial t}\frac{\partial^2R}{\partial r^2}}{135R^2}+\frac{e^{-2\lambda}\frac{\partial R}{\partial t}\frac{\partial^3R}{\partial r^3}}{2160R^2}$$

$$+\frac{e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial t}\frac{\partial R}{\partial r}\frac{\partial^2R}{\partial t^2}}{720R^2}-\frac{13e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial r}\frac{\partial R}{\partial t}\frac{\partial^2R}{\partial t^2}}{1080R^2}-\frac{e^{-\lambda-2\varphi}\frac{\partial R}{\partial r}\frac{\partial^3R}{\partial t^3}}{2160R^2}-\frac{e^{-\lambda-2\varphi}\left(\frac{\partial R}{\partial t}\right)^2\frac{\partial^2\lambda}{\partial r\,\partial t}}{4320R^2}$$

$$+\frac{e^{-2\lambda}\left(\frac{\partial R}{\partial r}\right)^2\frac{\partial^2\varphi}{\partial r\,\partial t}}{2160R^2}+\frac{e^{-2\lambda}\frac{\partial\lambda}{\partial r}\frac{\partial R}{\partial r}\frac{\partial^2R}{\partial r\,\partial t}}{135R^2}-\frac{29e^{-2\lambda}\frac{\partial\varphi}{\partial r}\frac{\partial R}{\partial r}\frac{\partial^2R}{\partial r\,\partial t}}{2160R^2}-\frac{13e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial t}\frac{\partial R}{\partial t}\frac{\partial^2R}{\partial r\,\partial t}}{1080R^2}$$

$$-\frac{29e^{-2\lambda}\frac{\partial^2R}{\partial r^2}\frac{\partial^2R}{\partial r\,\partial t}}{2160R^2}+\frac{29e^{-\lambda-2\varphi}\frac{\partial^2R}{\partial t^2}\frac{\partial^2R}{\partial r\,\partial t}}{2160R^2}+\frac{e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial r}\left(\frac{\partial R}{\partial t}\right)^3}{90R^3}-\frac{e^{-2\lambda}\frac{\partial\varphi}{\partial r}\left(\frac{\partial R}{\partial r}\right)^2\frac{\partial R}{\partial t}}{90R^3}$$

$$+\frac{5e^{-\lambda}\frac{\partial\varphi}{\partial r}\frac{\partial R}{\partial t}}{432R^3}-\frac{5e^{-\lambda}\frac{\partial^2R}{\partial r\,\partial t}}{432R^3}+\frac{e^{-2\lambda}\left(\frac{\partial R}{\partial r}\right)^2\frac{\partial^2R}{\partial r\,\partial t}}{90R^3}-\frac{e^{-\lambda-2\varphi}\left(\frac{\partial R}{\partial t}\right)^2\frac{\partial^2R}{\partial r\,\partial t}}{90R^3}$$

$$T_r{}^r=-\frac{e^{-4\varphi}\left(\frac{\partial\lambda}{\partial t}\right)^4}{11520}+\frac{e^{-4\varphi}\frac{\partial\varphi}{\partial t}\left(\frac{\partial\lambda}{\partial t}\right)^3}{3840}-\frac{31e^{-4\varphi}\frac{\partial R}{\partial t}\left(\frac{\partial\lambda}{\partial t}\right)^3}{11520R}+\frac{e^{-\lambda-2\varphi}\left(\frac{\partial\varphi}{\partial r}\right)^2\left(\frac{\partial\lambda}{\partial t}\right)^2}{4320}$$

$$+\frac{e^{-4\varphi}\left(\frac{\partial\varphi}{\partial t}\right)^2\left(\frac{\partial\lambda}{\partial t}\right)^2}{1440}-\frac{e^{-4\varphi}\frac{\partial^2\lambda}{\partial t^2}\left(\frac{\partial\lambda}{\partial t}\right)^2}{3840}-\frac{e^{-\lambda-2\varphi}\frac{\partial\lambda}{\partial r}\frac{\partial\varphi}{\partial r}\left(\frac{\partial\lambda}{\partial t}\right)^2}{4320}+\frac{e^{-\lambda-2\varphi}\frac{\partial^2\varphi}{\partial r^2}\left(\frac{\partial\lambda}{\partial t}\right)^2}{2160}$$

$$
\begin{aligned}
&-\frac{13e^{-4\varphi}\frac{\partial^2\varphi}{\partial t^2}\left(\frac{\partial\lambda}{\partial t}\right)^2}{34560}-\frac{17e^{-\lambda-2\varphi}\frac{\partial\lambda}{\partial r}\frac{\partial R}{\partial r}\left(\frac{\partial\lambda}{\partial t}\right)^2}{6912R}-\frac{7e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial r}\frac{\partial R}{\partial r}\left(\frac{\partial\lambda}{\partial t}\right)^2}{1920R}+\frac{e^{-4\varphi}\frac{\partial\varphi}{\partial t}\frac{\partial R}{\partial t}\left(\frac{\partial\lambda}{\partial t}\right)^2}{1440R}\\
&+\frac{79e^{-\lambda-2\varphi}\frac{\partial^2 R}{\partial r^2}\left(\frac{\partial\lambda}{\partial t}\right)^2}{17280R}+\frac{31e^{-4\varphi}\frac{\partial^2 R}{\partial t^2}\left(\frac{\partial\lambda}{\partial t}\right)^2}{8640R}+\frac{13e^{-\lambda-2\varphi}\left(\frac{\partial R}{\partial r}\right)^2\left(\frac{\partial\lambda}{\partial t}\right)^2}{1920R^2}-\frac{13e^{-4\varphi}\left(\frac{\partial R}{\partial t}\right)^2\left(\frac{\partial\lambda}{\partial t}\right)^2}{3840R^2}\\
&-\frac{1}{576}e^{-4\varphi}\left(\frac{\partial\varphi}{\partial t}\right)^3\frac{\partial\lambda}{\partial t}+\frac{13e^{-4\varphi}\frac{\partial^3\lambda}{\partial t^3}\frac{\partial\lambda}{\partial t}}{34560}-\frac{1}{540}e^{-\lambda-2\varphi}\left(\frac{\partial\varphi}{\partial r}\right)^2\frac{\partial\varphi}{\partial t}\frac{\partial\lambda}{\partial t}-\frac{7e^{-4\varphi}\frac{\partial^2\lambda}{\partial t^2}\frac{\partial\varphi}{\partial t}\frac{\partial\lambda}{\partial t}}{6912}\\
&+\frac{1}{864}e^{-\lambda-2\varphi}\frac{\partial\lambda}{\partial r}\frac{\partial\varphi}{\partial r}\frac{\partial\varphi}{\partial t}\frac{\partial\lambda}{\partial t}-\frac{1}{432}e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial t}\frac{\partial^2\varphi}{\partial r^2}\frac{\partial\lambda}{\partial t}+\frac{7e^{-4\varphi}\frac{\partial\varphi}{\partial t}\frac{\partial^2\varphi}{\partial t^2}\frac{\partial\lambda}{\partial t}}{3456}-\frac{e^{-4\varphi}\frac{\partial^3\varphi}{\partial t^3}\frac{\partial\lambda}{\partial t}}{3456}\\
&+\frac{e^{-\lambda-2\varphi}\frac{\partial\lambda}{\partial r}\frac{\partial^2\lambda}{\partial r\,\partial t}\frac{\partial\lambda}{\partial t}}{11520}-\frac{e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial r}\frac{\partial^2\lambda}{\partial r\,\partial t}\frac{\partial\lambda}{\partial t}}{11520}-\frac{e^{-\lambda-2\varphi}\frac{\partial^3\lambda}{\partial r^2\,\partial t}\frac{\partial\lambda}{\partial t}}{5760}-\frac{e^{-\lambda-2\varphi}\frac{\partial\lambda}{\partial r}\frac{\partial^2\varphi}{\partial r\,\partial t}\frac{\partial\lambda}{\partial t}}{1728}\\
&+\frac{1}{640}e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial r}\frac{\partial^2\varphi}{\partial r\,\partial t}\frac{\partial\lambda}{\partial t}+\frac{1}{864}e^{-\lambda-2\varphi}\frac{\partial^3\varphi}{\partial r^2\,\partial t}\frac{\partial\lambda}{\partial t}+\frac{97e^{-\lambda-2\varphi}\frac{\partial\lambda}{\partial r}\frac{\partial\varphi}{\partial t}\frac{\partial R}{\partial r}\frac{\partial\lambda}{\partial t}}{17280R}\\
&+\frac{13e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial r}\frac{\partial\varphi}{\partial t}\frac{\partial R}{\partial r}\frac{\partial\lambda}{\partial t}}{2880R}+\frac{101e^{-\lambda-2\varphi}\left(\frac{\partial\varphi}{\partial r}\right)^2\frac{\partial R}{\partial t}\frac{\partial\lambda}{\partial t}}{8640R}+\frac{11e^{-4\varphi}\left(\frac{\partial\varphi}{\partial t}\right)^2\frac{\partial R}{\partial t}\frac{\partial\lambda}{\partial t}}{960R}\\
&-\frac{37e^{-4\varphi}\frac{\partial^2\lambda}{\partial t^2}\frac{\partial R}{\partial t}\frac{\partial\lambda}{\partial t}}{8640R}-\frac{109e^{-\lambda-2\varphi}\frac{\partial\lambda}{\partial r}\frac{\partial\varphi}{\partial r}\frac{\partial R}{\partial t}\frac{\partial\lambda}{\partial t}}{17280R}+\frac{103e^{-\lambda-2\varphi}\frac{\partial^2\varphi}{\partial r^2}\frac{\partial R}{\partial t}\frac{\partial\lambda}{\partial t}}{8640R}\\
&-\frac{e^{-4\varphi}\frac{\partial^2\varphi}{\partial t^2}\frac{\partial R}{\partial t}\frac{\partial\lambda}{\partial t}}{480R}-\frac{97e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial t}\frac{\partial^2 R}{\partial r^2}\frac{\partial\lambda}{\partial t}}{8640R}-\frac{29e^{-4\varphi}\frac{\partial\varphi}{\partial t}\frac{\partial^2 R}{\partial t^2}\frac{\partial\lambda}{\partial t}}{2880R}+\frac{e^{-4\varphi}\frac{\partial^3 R}{\partial t^3}\frac{\partial\lambda}{\partial t}}{1080R}\\
&-\frac{e^{-\lambda-2\varphi}\frac{\partial R}{\partial r}\frac{\partial^2\lambda}{\partial r\,\partial t}\frac{\partial\lambda}{\partial t}}{640R}+\frac{11e^{-\lambda-2\varphi}\frac{\partial R}{\partial r}\frac{\partial^2\varphi}{\partial r\,\partial t}\frac{\partial\lambda}{\partial t}}{4320R}-\frac{e^{-\lambda-2\varphi}\frac{\partial\lambda}{\partial r}\frac{\partial^2 R}{\partial r\,\partial t}\frac{\partial\lambda}{\partial t}}{2160R}+\frac{e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial r}\frac{\partial^2 R}{\partial r\,\partial t}\frac{\partial\lambda}{\partial t}}{360R}\\
&+\frac{7e^{-\lambda-2\varphi}\frac{\partial^3 R}{\partial r^2\,\partial t}\frac{\partial\lambda}{\partial t}}{4320R}-\frac{11e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial t}\left(\frac{\partial R}{\partial r}\right)^2\frac{\partial\lambda}{\partial t}}{8640R^2}-\frac{e^{-4\varphi}\frac{\partial\varphi}{\partial t}\left(\frac{\partial R}{\partial t}\right)^2\frac{\partial\lambda}{\partial t}}{180R^2}-\frac{13e^{-\lambda-2\varphi}\frac{\partial\lambda}{\partial r}\frac{\partial R}{\partial r}\frac{\partial R}{\partial t}\frac{\partial\lambda}{\partial t}}{1920R^2}\\
&+\frac{35e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial r}\frac{\partial R}{\partial r}\frac{\partial R}{\partial t}\frac{\partial\lambda}{\partial t}}{1728R^2}+\frac{13e^{-\lambda-2\varphi}\frac{\partial R}{\partial t}\frac{\partial^2 R}{\partial r^2}\frac{\partial\lambda}{\partial t}}{960R^2}+\frac{e^{-4\varphi}\frac{\partial R}{\partial t}\frac{\partial^2 R}{\partial t^2}\frac{\partial\lambda}{\partial t}}{180R^2}-\frac{211e^{-\lambda-2\varphi}\frac{\partial R}{\partial r}\frac{\partial^2 R}{\partial r\,\partial t}\frac{\partial\lambda}{\partial t}}{8640R^2}\\
&+\frac{5e^{-4\varphi}\left(\frac{\partial R}{\partial t}\right)^3\frac{\partial\lambda}{\partial t}}{864R^3}+\frac{5e^{-2\varphi}\frac{\partial R}{\partial t}\frac{\partial\lambda}{\partial t}}{864R^3}-\frac{31e^{-\lambda-2\varphi}\left(\frac{\partial R}{\partial r}\right)^2\frac{\partial R}{\partial t}\frac{\partial\lambda}{\partial t}}{4320R^3}-\frac{1}{720}e^{-2\lambda}\left(\frac{\partial\varphi}{\partial r}\right)^4\\
&+\frac{7e^{-2\lambda}\frac{\partial\lambda}{\partial r}\left(\frac{\partial\varphi}{\partial r}\right)^3}{8640}-\frac{e^{-4\varphi}\left(\frac{\partial^2\lambda}{\partial t^2}\right)^2}{17280}+\frac{e^{-2\lambda}\left(\frac{\partial\lambda}{\partial r}\right)^2\left(\frac{\partial\varphi}{\partial r}\right)^2}{2160}-\frac{11e^{-2\lambda}\frac{\partial^2\lambda}{\partial r^2}\left(\frac{\partial\varphi}{\partial r}\right)^2}{17280}\\
&+\frac{1}{540}e^{-\lambda-2\varphi}\frac{\partial^2\lambda}{\partial t^2}\left(\frac{\partial\varphi}{\partial r}\right)^2+\frac{11e^{-4\varphi}\frac{\partial^2\lambda}{\partial t^2}\left(\frac{\partial\varphi}{\partial t}\right)^2}{3456}-\frac{e^{-2\lambda}\left(\frac{\partial^2\varphi}{\partial r^2}\right)^2}{1440}-\frac{e^{-\lambda-2\varphi}\left(\frac{\partial^2\lambda}{\partial r\,\partial t}\right)^2}{5760}\\
&-\frac{1}{864}e^{-\lambda-2\varphi}\left(\frac{\partial^2\varphi}{\partial r\,\partial t}\right)^2+\frac{e^{-4\varphi}\frac{\partial^4\lambda}{\partial t^4}}{3456}-\frac{e^{-2\lambda}\left(\frac{\partial\lambda}{\partial r}\right)^3\frac{\partial\varphi}{\partial r}}{3840}+\frac{7e^{-2\lambda}\frac{\partial\lambda}{\partial r}\frac{\partial^2\lambda}{\partial r^2}\frac{\partial\varphi}{\partial r}}{11520}-\frac{e^{-2\lambda}\frac{\partial^3\lambda}{\partial r^3}\frac{\partial\varphi}{\partial r}}{5760}\\
&-\frac{1}{864}e^{-\lambda-2\varphi}\frac{\partial\lambda}{\partial r}\frac{\partial^2\lambda}{\partial t^2}\frac{\partial\varphi}{\partial r}-\frac{1}{576}e^{-4\varphi}\frac{\partial^3\lambda}{\partial t^3}\frac{\partial\varphi}{\partial t}+\frac{11e^{-2\lambda}\left(\frac{\partial\lambda}{\partial r}\right)^2\frac{\partial^2\varphi}{\partial r^2}}{11520}-\frac{7e^{-2\lambda}\left(\frac{\partial\varphi}{\partial r}\right)^2\frac{\partial^2\varphi}{\partial r^2}}{4320}\\
&-\frac{e^{-2\lambda}\frac{\partial^2\lambda}{\partial r^2}\frac{\partial^2\varphi}{\partial r^2}}{1440}+\frac{1}{432}e^{-\lambda-2\varphi}\frac{\partial^2\lambda}{\partial t^2}\frac{\partial^2\varphi}{\partial r^2}-\frac{7e^{-2\lambda}\frac{\partial\lambda}{\partial r}\frac{\partial\varphi}{\partial r}\frac{\partial^2\varphi}{\partial r^2}}{5760}-\frac{1}{960}e^{-2\lambda}\frac{\partial\lambda}{\partial r}\frac{\partial^3\varphi}{\partial r^3}+\frac{11e^{-2\lambda}\frac{\partial\varphi}{\partial r}\frac{\partial^3\varphi}{\partial r^3}}{8640}\\
&+\frac{e^{-2\lambda}\frac{\partial^4\varphi}{\partial r^4}}{2880}-\frac{1}{864}e^{-4\varphi}\frac{\partial^2\lambda}{\partial t^2}\frac{\partial^2\varphi}{\partial t^2}-\frac{e^{-\lambda-2\varphi}\frac{\partial\lambda}{\partial r}\frac{\partial\varphi}{\partial t}\frac{\partial^2\lambda}{\partial r\,\partial t}}{11520}-\frac{e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial r}\frac{\partial\varphi}{\partial t}\frac{\partial^2\lambda}{\partial r\,\partial t}}{1440}\\
&+\frac{e^{-\lambda-2\varphi}\frac{\partial\lambda}{\partial r}\frac{\partial^3\lambda}{\partial r\,\partial t^2}}{11520}+\frac{e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial r}\frac{\partial^3\lambda}{\partial r\,\partial t^2}}{1440}+\frac{e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial t}\frac{\partial^3\lambda}{\partial r^2\,\partial t}}{5760}-\frac{e^{-\lambda-2\varphi}\frac{\partial^4\lambda}{\partial r^2\,\partial t^2}}{5760}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{e^{-\lambda-2\varphi}\frac{\partial\lambda}{\partial r}\frac{\partial\varphi}{\partial t}\frac{\partial^2\varphi}{\partial r\,\partial t}}{3456}+\frac{1}{864}e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial r}\frac{\partial\varphi}{\partial t}\frac{\partial^2\varphi}{\partial r\,\partial t}+\frac{e^{-\lambda-2\varphi}\frac{\partial^2\lambda}{\partial r\,\partial t}\frac{\partial^2\varphi}{\partial r\,\partial t}}{1080}+\frac{e^{-\lambda-2\varphi}\frac{\partial\lambda}{\partial r}\frac{\partial^3\varphi}{\partial r\,\partial t^2}}{3456}\\
&-\frac{1}{864}e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial r}\frac{\partial^3\varphi}{\partial r\,\partial t^2}+\frac{e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial t}\frac{\partial^3\varphi}{\partial r^2\,\partial t}}{1728}-\frac{e^{-\lambda-2\varphi}\frac{\partial^4\varphi}{\partial r^2\,\partial t^2}}{1728}-\frac{e^{-2\lambda}\left(\frac{\partial\lambda}{\partial r}\right)^3\frac{\partial R}{\partial r}}{1920R}\\
&+\frac{59e^{-2\lambda}\left(\frac{\partial\varphi}{\partial r}\right)^3\frac{\partial R}{\partial r}}{4320R}+\frac{e^{-2\lambda}\frac{\partial\lambda}{\partial r}\left(\frac{\partial\varphi}{\partial r}\right)^2\frac{\partial R}{\partial r}}{432R}+\frac{7e^{-2\lambda}\frac{\partial\lambda}{\partial r}\frac{\partial^2\lambda}{\partial r^2}\frac{\partial R}{\partial r}}{5760R}-\frac{e^{-2\lambda}\frac{\partial^3\lambda}{\partial r^3}\frac{\partial R}{\partial r}}{2880R}\\
&-\frac{97e^{-\lambda-2\varphi}\frac{\partial\lambda}{\partial r}\frac{\partial^2\lambda}{\partial t^2}\frac{\partial R}{\partial r}}{17280R}-\frac{49e^{-2\lambda}\left(\frac{\partial\lambda}{\partial r}\right)^2\frac{\partial\varphi}{\partial r}\frac{\partial R}{\partial r}}{17280R}-\frac{e^{-2\lambda}\frac{\partial^2\lambda}{\partial r^2}\frac{\partial\varphi}{\partial r}\frac{\partial R}{\partial r}}{540R}-\frac{11e^{-\lambda-2\varphi}\frac{\partial^2\lambda}{\partial t^2}\frac{\partial\varphi}{\partial r}\frac{\partial R}{\partial r}}{2880R}\\
&+\frac{19e^{-2\lambda}\frac{\partial\lambda}{\partial r}\frac{\partial^2\varphi}{\partial r^2}\frac{\partial R}{\partial r}}{2880R}+\frac{17e^{-2\lambda}\frac{\partial\varphi}{\partial r}\frac{\partial^2\varphi}{\partial r^2}\frac{\partial R}{\partial r}}{1080R}+\frac{e^{-2\lambda}\frac{\partial^3\varphi}{\partial r^3}\frac{\partial R}{\partial r}}{540R}-\frac{e^{-4\varphi}\left(\frac{\partial\varphi}{\partial t}\right)^3\frac{\partial R}{\partial t}}{144R}+\frac{e^{-4\varphi}\frac{\partial^3\lambda}{\partial t^3}\frac{\partial R}{\partial t}}{864R}\\
&+\frac{67e^{-\lambda-2\varphi}\left(\frac{\partial\varphi}{\partial r}\right)^2\frac{\partial\varphi}{\partial t}\frac{\partial R}{\partial t}}{4320R}-\frac{31e^{-4\varphi}\frac{\partial^2\lambda}{\partial t^2}\frac{\partial\varphi}{\partial t}\frac{\partial R}{\partial t}}{2880R}-\frac{47e^{-\lambda-2\varphi}\frac{\partial\lambda}{\partial r}\frac{\partial\varphi}{\partial r}\frac{\partial\varphi}{\partial t}\frac{\partial R}{\partial t}}{8640R}+\frac{47e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial t}\frac{\partial^2\varphi}{\partial r^2}\frac{\partial R}{\partial t}}{4320R}\\
&+\frac{7e^{-4\varphi}\frac{\partial\varphi}{\partial t}\frac{\partial^2\varphi}{\partial t^2}\frac{\partial R}{\partial t}}{864R}-\frac{e^{-4\varphi}\frac{\partial^3\varphi}{\partial t^3}\frac{\partial R}{\partial t}}{864R}+\frac{11e^{-2\lambda}\left(\frac{\partial\lambda}{\partial r}\right)^2\frac{\partial^2R}{\partial r^2}}{5760R}-\frac{11e^{-2\lambda}\left(\frac{\partial\varphi}{\partial r}\right)^2\frac{\partial^2R}{\partial r^2}}{540R}\\
&-\frac{e^{-2\lambda}\frac{\partial^2\lambda}{\partial r^2}\frac{\partial^2R}{\partial r^2}}{720R}+\frac{97e^{-\lambda-2\varphi}\frac{\partial^2\lambda}{\partial t^2}\frac{\partial^2R}{\partial r^2}}{8640R}+\frac{19e^{-2\lambda}\frac{\partial\lambda}{\partial r}\frac{\partial\varphi}{\partial r}\frac{\partial^2R}{\partial r^2}}{2880R}-\frac{3e^{-2\lambda}\frac{\partial^2\varphi}{\partial r^2}\frac{\partial^2R}{\partial r^2}}{160R}-\frac{e^{-2\lambda}\frac{\partial\lambda}{\partial r}\frac{\partial^3R}{\partial r^3}}{480R}\\
&+\frac{e^{-2\lambda}\frac{\partial\varphi}{\partial r}\frac{\partial^3R}{\partial r^3}}{540R}+\frac{e^{-2\lambda}\frac{\partial^4R}{\partial r^4}}{1440R}-\frac{61e^{-\lambda-2\varphi}\left(\frac{\partial\varphi}{\partial r}\right)^2\frac{\partial^2R}{\partial t^2}}{4320R}+\frac{11e^{-4\varphi}\left(\frac{\partial\varphi}{\partial t}\right)^2\frac{\partial^2R}{\partial t^2}}{864R}+\frac{7e^{-4\varphi}\frac{\partial^2\lambda}{\partial t^2}\frac{\partial^2R}{\partial t^2}}{960R}\\
&+\frac{47e^{-\lambda-2\varphi}\frac{\partial\lambda}{\partial r}\frac{\partial\varphi}{\partial r}\frac{\partial^2R}{\partial t^2}}{8640R}-\frac{47e^{-\lambda-2\varphi}\frac{\partial^2\varphi}{\partial r^2}\frac{\partial^2R}{\partial t^2}}{4320R}-\frac{e^{-4\varphi}\frac{\partial^2\varphi}{\partial t^2}\frac{\partial^2R}{\partial t^2}}{216R}-\frac{e^{-4\varphi}\frac{\partial\varphi}{\partial t}\frac{\partial^3R}{\partial t^3}}{144R}+\frac{e^{-4\varphi}\frac{\partial^4R}{\partial t^4}}{864R}\\
&+\frac{e^{-\lambda-2\varphi}\frac{\partial\lambda}{\partial r}\frac{\partial R}{\partial t}\frac{\partial^2\lambda}{\partial r\,\partial t}}{5760R}+\frac{e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial r}\frac{\partial R}{\partial t}\frac{\partial^2\lambda}{\partial r\,\partial t}}{720R}-\frac{e^{-\lambda-2\varphi}\frac{\partial R}{\partial t}\frac{\partial^3\lambda}{\partial r^2\,\partial t}}{2880R}+\frac{e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial t}\frac{\partial R}{\partial r}\frac{\partial^2\varphi}{\partial r\,\partial t}}{864R}\\
&+\frac{e^{-\lambda-2\varphi}\frac{\partial\lambda}{\partial r}\frac{\partial R}{\partial t}\frac{\partial^2\varphi}{\partial r\,\partial t}}{4320R}-\frac{17e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial r}\frac{\partial R}{\partial t}\frac{\partial^2\varphi}{\partial r\,\partial t}}{4320R}-\frac{e^{-\lambda-2\varphi}\frac{\partial R}{\partial r}\frac{\partial^3\varphi}{\partial r\,\partial t^2}}{864R}-\frac{e^{-\lambda-2\varphi}\frac{\partial R}{\partial t}\frac{\partial^3\varphi}{\partial r^2\,\partial t}}{2160R}\\
&-\frac{e^{-\lambda-2\varphi}\frac{\partial\lambda}{\partial r}\frac{\partial\varphi}{\partial t}\frac{\partial^2R}{\partial r\,\partial t}}{1080R}-\frac{e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial r}\frac{\partial\varphi}{\partial t}\frac{\partial^2R}{\partial r\,\partial t}}{540R}+\frac{e^{-\lambda-2\varphi}\frac{\partial^2\lambda}{\partial r\,\partial t}\frac{\partial^2R}{\partial r\,\partial t}}{2160R}-\frac{e^{-\lambda-2\varphi}\frac{\partial^2\varphi}{\partial r\,\partial t}\frac{\partial^2R}{\partial r\,\partial t}}{1080R}\\
&+\frac{e^{-\lambda-2\varphi}\frac{\partial\lambda}{\partial r}\frac{\partial^3R}{\partial r\,\partial t^2}}{1080R}+\frac{e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial r}\frac{\partial^3R}{\partial r\,\partial t^2}}{2160R}+\frac{e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial t}\frac{\partial^3R}{\partial r^2\,\partial t}}{540R}-\frac{e^{-\lambda-2\varphi}\frac{\partial^4R}{\partial r^2\,\partial t^2}}{540R}\\
&-\frac{83e^{-2\lambda}\left(\frac{\partial\lambda}{\partial r}\right)^2\left(\frac{\partial R}{\partial r}\right)^2}{34560R^2}-\frac{17e^{-2\lambda}\left(\frac{\partial\varphi}{\partial r}\right)^2\left(\frac{\partial R}{\partial r}\right)^2}{8640R^2}-\frac{7e^{-2\lambda}\frac{\partial^2\lambda}{\partial r^2}\left(\frac{\partial R}{\partial r}\right)^2}{8640R^2}+\frac{11e^{-\lambda-2\varphi}\frac{\partial^2\lambda}{\partial t^2}\left(\frac{\partial R}{\partial r}\right)^2}{8640R^2}\\
&-\frac{13e^{-2\lambda}\frac{\partial\lambda}{\partial r}\frac{\partial\varphi}{\partial r}\left(\frac{\partial R}{\partial r}\right)^2}{2160R^2}-\frac{e^{-2\lambda}\frac{\partial^2\varphi}{\partial r^2}\left(\frac{\partial R}{\partial r}\right)^2}{2160R^2}+\frac{31e^{-\lambda-2\varphi}\left(\frac{\partial\varphi}{\partial r}\right)^2\left(\frac{\partial R}{\partial t}\right)^2}{1440R^2}+\frac{5e^{-4\varphi}\left(\frac{\partial\varphi}{\partial t}\right)^2\left(\frac{\partial R}{\partial t}\right)^2}{1728R^2}\\
&-\frac{e^{-\lambda-2\varphi}\frac{\partial\lambda}{\partial r}\frac{\partial\varphi}{\partial r}\left(\frac{\partial R}{\partial t}\right)^2}{2880R^2}+\frac{e^{-\lambda-2\varphi}\frac{\partial^2\varphi}{\partial r^2}\left(\frac{\partial R}{\partial t}\right)^2}{1440R^2}-\frac{e^{-4\varphi}\frac{\partial^2\varphi}{\partial t^2}\left(\frac{\partial R}{\partial t}\right)^2}{864R^2}-\frac{37e^{-2\lambda}\left(\frac{\partial^2R}{\partial r^2}\right)^2}{2880R^2}\\
&+\frac{e^{-4\varphi}\left(\frac{\partial^2R}{\partial t^2}\right)^2}{1728R^2}+\frac{49e^{-\lambda-2\varphi}\left(\frac{\partial^2R}{\partial r\,\partial t}\right)^2}{2160R^2}-\frac{e^{-\lambda-2\varphi}\frac{\partial\lambda}{\partial r}\frac{\partial\varphi}{\partial t}\frac{\partial R}{\partial r}\frac{\partial R}{\partial t}}{192R^2}-\frac{13e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial r}\frac{\partial\varphi}{\partial t}\frac{\partial R}{\partial r}\frac{\partial R}{\partial t}}{4320R^2}\\
&+\frac{e^{-2\lambda}\frac{\partial\lambda}{\partial r}\frac{\partial R}{\partial r}\frac{\partial^2R}{\partial r^2}}{96R^2}+\frac{e^{-2\lambda}\frac{\partial\varphi}{\partial r}\frac{\partial R}{\partial r}\frac{\partial^2R}{\partial r^2}}{80R^2}+\frac{e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial t}\frac{\partial R}{\partial t}\frac{\partial^2R}{\partial r^2}}{96R^2}+\frac{7e^{-2\lambda}\frac{\partial R}{\partial r}\frac{\partial^3R}{\partial r^3}}{4320R^2}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{e^{-\lambda-2\varphi}\frac{\partial\lambda}{\partial r}\frac{\partial R}{\partial r}\frac{\partial^2 R}{\partial t^2}}{192R^2}+\frac{13e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial r}\frac{\partial R}{\partial r}\frac{\partial^2 R}{\partial t^2}}{4320R^2}-\frac{e^{-4\varphi}\frac{\partial\varphi}{\partial t}\frac{\partial R}{\partial t}\frac{\partial^2 R}{\partial t^2}}{216R^2}-\frac{e^{-\lambda-2\varphi}\frac{\partial^2 R}{\partial r^2}\frac{\partial^2 R}{\partial t^2}}{96R^2}\\
&+\frac{e^{-4\varphi}\frac{\partial R}{\partial t}\frac{\partial^3 R}{\partial t^3}}{864R^2}-\frac{e^{-\lambda-2\varphi}\frac{\partial R}{\partial r}\frac{\partial R}{\partial t}\frac{\partial^2\lambda}{\partial r\,\partial t}}{2160R^2}+\frac{e^{-\lambda-2\varphi}\frac{\partial R}{\partial r}\frac{\partial R}{\partial t}\frac{\partial^2\varphi}{\partial r\,\partial t}}{1080R^2}+\frac{e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial t}\frac{\partial R}{\partial r}\frac{\partial^2 R}{\partial r\,\partial t}}{480R^2}\\
&+\frac{e^{-\lambda-2\varphi}\frac{\partial\lambda}{\partial r}\frac{\partial R}{\partial t}\frac{\partial^2 R}{\partial r\,\partial t}}{2880R^2}-\frac{193e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial r}\frac{\partial R}{\partial t}\frac{\partial^2 R}{\partial r\,\partial t}}{4320R^2}-\frac{e^{-\lambda-2\varphi}\frac{\partial R}{\partial r}\frac{\partial^3 R}{\partial r\,\partial t^2}}{480R^2}-\frac{e^{-\lambda-2\varphi}\frac{\partial R}{\partial t}\frac{\partial^3 R}{\partial r^2\,\partial t}}{1440R^2}\\
&-\frac{19e^{-2\lambda}\frac{\partial\lambda}{\partial r}\left(\frac{\partial R}{\partial r}\right)^3}{4320R^3}+\frac{e^{-2\lambda}\frac{\partial\varphi}{\partial r}\left(\frac{\partial R}{\partial r}\right)^3}{2160R^3}+\frac{e^{-4\varphi}\frac{\partial\varphi}{\partial t}\left(\frac{\partial R}{\partial t}\right)^3}{432R^3}+\frac{5e^{-\lambda-2\varphi}\frac{\partial\lambda}{\partial r}\frac{\partial R}{\partial r}\left(\frac{\partial R}{\partial t}\right)^2}{864R^3}\\
&-\frac{7e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial r}\frac{\partial R}{\partial r}\left(\frac{\partial R}{\partial t}\right)^2}{2160R^3}+\frac{5e^{-\lambda}\frac{\partial\lambda}{\partial r}\frac{\partial R}{\partial r}}{864R^3}+\frac{e^{-\lambda-2\varphi}\frac{\partial\varphi}{\partial t}\left(\frac{\partial R}{\partial r}\right)^2\frac{\partial R}{\partial t}}{2160R^3}-\frac{5e^{-\lambda}\frac{\partial^2 R}{\partial r^2}}{432R^3}\\
&+\frac{19e^{-2\lambda}\left(\frac{\partial R}{\partial r}\right)^2\frac{\partial^2 R}{\partial r^2}}{2160R^3}-\frac{5e^{-\lambda-2\varphi}\left(\frac{\partial R}{\partial t}\right)^2\frac{\partial^2 R}{\partial r^2}}{432R^3}-\frac{e^{-\lambda-2\varphi}\left(\frac{\partial R}{\partial r}\right)^2\frac{\partial^2 R}{\partial t^2}}{2160R^3}-\frac{e^{-4\varphi}\left(\frac{\partial R}{\partial t}\right)^2\frac{\partial^2 R}{\partial t^2}}{432R^3}\\
&+\frac{e^{-\lambda-2\varphi}\frac{\partial R}{\partial r}\frac{\partial R}{\partial t}\frac{\partial^2 R}{\partial r\,\partial t}}{180R^3}+\frac{e^{-2\lambda}\left(\frac{\partial R}{\partial r}\right)^4}{864R^4}+\frac{e^{-4\varphi}\left(\frac{\partial R}{\partial t}\right)^4}{864R^4}-\frac{e^{-\lambda}\left(\frac{\partial R}{\partial r}\right)^2}{864R^4}+\frac{e^{-2\varphi}\left(\frac{\partial R}{\partial t}\right)^2}{864R^4}\\
&-\frac{e^{-\lambda-2\varphi}\left(\frac{\partial R}{\partial r}\right)^2\left(\frac{\partial R}{\partial t}\right)^2}{432R^4}
\end{aligned}
$$

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